So we need $\Gamma^0_{\alpha\beta} \dfrac{p^\alpha p^\beta}{p}$

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]$$

$$\Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p} = \underbrace{\frac{g^{0\nu}}{2}\left[2\frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]\frac{p^\alpha p^\beta}{p}}_{\text{symmetric in } \alpha,\beta} = \underbrace{\frac{g^{00}}{2}\left[2\frac{\partial g_{0\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^0}\right]\frac{p^\alpha p^\beta}{p}}_{\nu=0}$$

$$g^{00} = (g_{00})^{-1} = \left[-(1+2\Psi)\right]^{-1} \simeq -1+2\Psi$$

check $(-1-2\Psi)(-1+2\Psi) =$
$= 1 - \cancel{2\Psi} + \cancel{2\Psi} - \cancel{4\Psi^2} \simeq 1$
2nd order

$$\frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}$$

$$\Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p} = \frac{-1+2\Psi}{2}\left[2\frac{\partial g_{0\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial t}\right]\frac{p^\alpha p^\beta}{p} = \frac{-1+2\Psi}{2}\left[-4\frac{\partial \Psi}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial t}\right]\frac{p^\alpha p^\beta}{p}$$

$\alpha = 0$, $\dfrac{\partial g_{0\alpha}}{\partial x^\beta} = \dfrac{\partial}{\partial x^\beta}\left[-1-2\Psi\right] \simeq -2\dfrac{\partial \Psi}{\partial x^\beta}$

$$-\frac{\partial g_{\alpha\beta}}{\partial t}\frac{p^\alpha p^\beta}{p} = -\frac{\partial g_{00}}{\partial t}\frac{p^0 p^0}{p} - \frac{\partial g_{ij}}{\partial t}\frac{p^i p^j}{p} = 2\frac{\partial \Psi}{\partial t}p - a^2\delta_{ij}\left[2\frac{\partial \Phi}{\partial t} + 2H(1+2\Phi)\right]\frac{p^i p^j}{p}$$

$\dfrac{\partial g_{00}}{\partial t} \simeq -2\dfrac{\partial \Psi}{\partial t}$, $\dfrac{(p^0)^2}{p} \simeq \dfrac{p^2(1-2\Psi)}{p} = p(1-2\Psi)$

$\to (-) -2\dfrac{\partial \Psi}{\partial t} p(1-2\Psi) \simeq 2\dfrac{\partial \Psi}{\partial t}p$

$$\frac{\partial g_{ij}}{\partial t} = \frac{\partial}{\partial t}\left[a^2\delta_{ij}(1+2\Phi)\right] =$$

$$= a^2\delta_{ij}\frac{\partial}{\partial t}\left[1+2\Phi\right] + (1+2\Phi)\delta_{ij}\frac{\partial}{\partial t}a^2 = a^2\delta_{ij}\frac{2\partial\Phi}{\partial t} + (1+2\Phi)\delta_{ij}\dot{a}\,2a =$$

$$= a^2\delta_{ij}\frac{2\partial\Phi}{\partial t} + a^2\delta_{ij}(1+2\Phi)\,2\underbrace{\frac{\dot a}{a}}_{H} = a^2\delta_{ij}\left[\frac{2\partial\Phi}{\partial t} + 2H(1+2\Phi)\right]$$

So

$$-\frac{\partial g_{\alpha\beta}}{\partial t}\frac{p^\alpha p^\beta}{p} \simeq 2\frac{\partial\Psi}{\partial t}p - a^2\delta_{ij}\left[2\frac{\partial\Phi}{\partial t} + 2H(1+2\Phi)\right]\frac{p^i p^j}{p} \simeq$$

$\dfrac{p^i p^j}{p}\delta_{ij} \simeq \dfrac{p^2}{a^2}(1-2\Phi)$

$\simeq \cancel{2\frac{\partial\psi}{\partial t}p - 2a^2\delta_{ij}\frac{\partial\Phi}{\partial t}}$

$(1+2\Phi)(1-2\Phi)\simeq 1$

$(1-2\Phi)\frac{\partial\Phi}{\partial t}\simeq\frac{\partial\Phi}{\partial t}$

$$\simeq 2\frac{\partial\psi}{\partial t}p - \cancel{a^2}\frac{p}{\cancel{a^2}}(1-2\Phi)\left[2\frac{\partial\Phi}{\partial t}+2H(1+2\Phi)\right] \simeq 2\frac{\partial\psi}{\partial t}p - 2p\frac{\partial\Phi}{\partial t} - 2Hp$$

Putting everything together

$$\Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p} = \frac{-1+2\psi}{2}\left[2\frac{\partial g_{0\alpha}}{\partial x^\beta}-\frac{\partial g_{\alpha\beta}}{\partial t}\right]\frac{p^\alpha p^\beta}{p} \simeq \frac{-1+2\psi}{2}\left[-4\frac{\partial\psi}{\partial x^\beta}p^\beta + 2p\frac{\partial\psi}{\partial t} - 2p\frac{\partial\Phi}{\partial t} - 2Hp\right] \simeq$$

$$2\frac{\partial g_{0\alpha}}{\partial x^\beta}\frac{p^\alpha p^\beta}{p} = -4\frac{\partial\psi}{\partial x^\beta}\frac{p^0 p^\beta}{p} \simeq -4\frac{\partial\psi}{\partial x^\beta}\frac{\cancel{p}(1-\psi)p^\beta}{\cancel{p}} \simeq -4\frac{\partial\psi}{\partial x^\beta}p^\beta \qquad \frac{\partial\psi}{\partial x^\beta}(1-\psi)\simeq\frac{\partial\psi}{\partial x^\beta}$$

$$\simeq \frac{-1+2\psi}{2}\left[-4\left(\frac{\partial\psi}{\partial t}p + \frac{\partial\psi}{\partial x^i}\frac{p\hat{p}^i}{a}\right) + 2p\frac{\partial\psi}{\partial t} - p\left(2\frac{\partial\Phi}{\partial t}+2H\right)\right] \simeq$$

$$\frac{\partial\psi}{\partial x^\beta}p^\beta \equiv \sum_\beta\frac{\partial\psi}{\partial x^\beta}p^\beta = \frac{\partial\psi}{\partial t}p^0 + \frac{\partial\psi}{\partial x^i}p^i \simeq \frac{\partial\psi}{\partial t}p(1-\psi) + \frac{\partial\psi}{\partial x^i}\frac{p\hat{p}^i}{a}(1-\Phi) \simeq \frac{\partial\psi}{\partial t}p + \frac{\partial\psi}{\partial x^i}\frac{p\hat{p}^i}{a}$$

$$\simeq \left(\frac{-1+2\psi}{2}\right)\left[-4p\frac{\partial\psi}{\partial t} - 4\frac{\partial\psi}{\partial x^i}\frac{p\hat{p}^i}{a} + 2p\frac{\partial\psi}{\partial t} - 2p\frac{\partial\Phi}{\partial t} - 2Hp\right] =$$

$$= \cancel{(-1+2\psi)} = (-1+2\psi)\left[-p\frac{\partial\psi}{\partial t} - 2\frac{\partial\psi}{\partial x^i}\frac{p\hat{p}^i}{a} - p\left(\frac{\partial\Phi}{\partial t}+H\right)\right]$$

Recall where we started

$$\frac{dp}{dt} = p\left(\frac{\partial\psi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\psi}{\partial x^i}\right) - \Gamma^0_{\alpha\beta}\frac{p^\alpha p^\beta}{p}(1+2\psi) = p\left(\cancel{\frac{\partial\psi}{\partial t}} + \frac{\hat{p}^i}{a}\frac{\partial\psi}{\partial x^i}\right) - \cancel{p\frac{\partial\psi}{\partial t}} - 2\frac{\partial\psi}{\partial x^i}\frac{p\hat{p}^i}{a} - p\left(\frac{\partial\Phi}{\partial t}+H\right)$$

$(-1+2\psi)(1+2\psi)\simeq -1$

$$\Rightarrow \boxed{\frac{1}{p}\frac{dp}{dt} = -H - \frac{\partial\Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial\psi}{\partial x^i}}$$

change in momentum of photon moving through perturbed FLRW Universe

Physical interpretation:

overdense region $\psi<0, \Phi>0$

- $-H$: loss of momentum due to Hubble expansion (redshift)
- $-\frac{\partial\Phi}{\partial t}>0$: γ in deepening gravitational well loses energy (redshift)
- $-\partial\psi/\partial x^i < 0$: γ travelling in a well gains energy, redshifted when leaving well

Big picture: return to full Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{dp}{dt}$$

$$\boxed{\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial t}}_{\text{(integrated) continuity equation}} + \underbrace{\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i}}_{\text{(integrated) Euler equation}} - p\frac{\partial f}{\partial p}\underbrace{\left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}\right]}_{\text{how photons lose energy in expanding Universe and in the presence of under/overdensities}} = 0}$$

(Collisionless) Boltzmann equations for photons

need to perturb and expand!

To make progress, expand $f(\bar{x}, \bar{p}, t) = f(\bar{x}, p, \hat{p}, t)$ around $f^{(0)}$ [LHS]

$$f^{(0)} = \frac{1}{\exp\left[\frac{p}{T(t)}\right] - 1} \quad \cancel{\rightarrow f(\bar{x}, p, \hat{p}, t) = \left[\exp\frac{p}{T(t)[1+\Theta(\bar{x},\hat{p},t)]}\right]}$$

$$\boxed{f = \left[\exp\left\{\frac{p}{T(t)[1+\Theta(\bar{x},\hat{p},t)]}\right\} - 1\right]^{-1}}$$

Zero-order temperature $T(t)$ function of time only $T \propto a^{-1}$

$\Theta = \frac{\delta T}{T}$ perturbation to the distribution function

$\Theta(\bar{x}, \hat{p}, t)$

- $\bar{x}$ → inhomogeneities
- $\hat{p}$ → anisotropies
- t → like zeroth-order temperature

no p dependence: see later! ~~Compton scattering leaves p unchanged~~

$\Theta \ll 1$, so expand in small Θ

useful property of Bose-Einstein distribution $T\frac{\partial f^{(0)}}{\partial T} = -p\frac{\partial f^{(0)}}{\partial p}$ easy to check!

Check: $f^{(0)} = \left[e^{\frac{p}{T}} - 1\right]^{-1}$

$$\frac{\partial f^{(0)}}{\partial T} = -\frac{1}{(e^{\frac{p}{T}}-1)^2} \times -\frac{p}{T^2} e^{\frac{p}{T}} = \frac{e^{p/T} p}{T^2 (e^{\frac{p}{T}}-1)^2}$$

$$\frac{\partial f^{(0)}}{\partial p} = -\frac{1}{(e^{\frac{p}{T}}-1)^2} \frac{1}{T} e^{\frac{p}{T}} = -\frac{e^{p/T}}{T(e^{\frac{p}{T}}-1)^2}$$

$$T \frac{\partial f^{(0)}}{\partial T} = -p \frac{\partial f^{(0)}}{\partial p}$$

Expand $f \simeq f^{(0)} + \delta f$ up to first order

$$f \simeq \left[e^{\frac{p}{T}} - 1\right]^{-1} + \frac{\partial f^{(0)}}{\partial T} T\Theta = f^{(0)} + \frac{\partial f^{(0)}}{\partial T} T\Theta = f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta$$

$$\boxed{f \simeq f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta}$$

perturbed distribution function

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial f}{\partial x^i} - p \frac{\partial f}{\partial p} \left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i}\right]$$

look at 0th and 1st order terms

0th order

~~EBSC~~. $$\frac{\partial f^{(0)}}{\partial t} + \underbrace{\cancel{\frac{\hat{p}^i}{a} \frac{\partial f^{(0)}}{\partial x^i}}}_{f^{(0)} \text{ doesn't depend on } \bar{x}} - p \frac{\partial f^{(0)}}{\partial p} \left[H + \underbrace{\cancel{\frac{\partial \Phi}{\partial t}}}_{\text{1st order}} + \underbrace{\cancel{\frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i}}}_{\text{1st order}}\right] = 0$$

$$\boxed{\frac{\partial f^{(0)}}{\partial t} - H p \frac{\partial f^{(0)}}{\partial p} = 0}$$

zeroth order

$\equiv$ (at zeroth order collisions vanish)

$$\frac{\partial f^{(0)}}{\partial t} - Hp\frac{\partial f^{(0)}}{\partial p} = 0$$

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial T}\frac{dT}{dt} = -\frac{dT}{dt}\frac{p}{T}\frac{\partial f^{(0)}}{\partial p} = \left(-\frac{dT/dt}{T}\right)p\frac{\partial f^{(0)}}{\partial p}$$

$$\rightarrow \left(-\frac{dT/dt}{T} - \frac{da/dt}{a}\right)p\frac{\partial f^{(0)}}{\partial p} \Longrightarrow \frac{1}{T}\frac{dT}{\cancel{dt}} = -\frac{1}{a}\frac{da}{\cancel{dt}} \Rightarrow \int\frac{dT}{T} = \int -\frac{da}{a}$$

$$\rightarrow \ln T = \ln(a^{-1}) + C \Longrightarrow \boxed{T \propto \frac{1}{a}}$$

Baltzmann equations at zeroth order confirm photon redshift !!!

<u>1st order</u>

$$\left.\frac{df}{dt}\right|_{1st\ order} = \frac{\partial}{\partial t}\Big[\underbrace{\cancel{f^{(0)}}}_{0th\ order} - p\frac{\partial f^{(0)}}{\partial p}\Theta\Big] + \frac{\hat{p}^i}{a}\frac{\partial}{\partial x^i}\Big[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta\Big] - Hp\frac{\partial}{\partial p}\Big[\underbrace{\cancel{f^{(0)}}}_{0th\ order} - p\frac{\partial f^{(0)}}{\partial p}\Theta\Big]$$

$$-p\frac{\partial\Phi}{\partial t}\frac{\partial}{\partial p}\Big[f^{(0)} - \underbrace{\cancel{p\frac{\partial f^{(0)}}{\partial p}\Theta}}_{2nd\ order}\Big] - \frac{p\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\frac{\partial}{\partial p}\Big[f^{(0)} - \underbrace{\cancel{p\frac{\partial f^{(0)}}{\partial p}\Theta}}_{2nd\ order}\Big]$$

1st order pieces

$$\left.\frac{df}{dt}\right|_{1st\ order} = -p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] - \frac{p\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}\frac{\partial f^{(0)}}{\partial p} + Hp\Theta\frac{\partial}{\partial p}\left[p\frac{\partial f^{(0)}}{\partial p}\right]$$

$\frac{\partial f^{(0)}}{\partial t}$ 0th order $\qquad \frac{\partial f^{(0)}}{\partial x^i} = 0 \qquad \frac{\partial f^{(0)}}{\partial p}$ 0th order

$$-p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]$$

$(\Phi,\Psi)\times\Theta$ 2nd order

$$\rightarrow -p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] = -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} - p\Theta\frac{\partial^2 f^{(0)}}{\partial t\partial p} = -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} - p\Theta\frac{dT}{dt}\frac{\partial^2 f^{(0)}}{\partial T\partial p} =$$

$$= -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial \Theta}{\partial t} + p\Theta\frac{dT/dt}{T}\frac{\partial}{\partial p}\left[p\frac{\partial f^{(0)}}{\partial p}\right]$$

$$\frac{\partial f^{(0)}}{\partial T} = -\frac{p}{T}\frac{\partial f^{(0)}}{\partial p}$$

$$\frac{1}{T}\frac{dT}{dt} = -\frac{1}{a}\frac{da}{dt} = -H$$

$$\left.\frac{df}{dt}\right|_{1\text{st order}} = -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} \cancel{+ p\Theta\frac{dT/dt}{T}\frac{\partial}{\partial p}\left[p\frac{\partial f^{(0)}}{\partial p}\right]} - \frac{p\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}\frac{\partial f^{(0)}}{\partial p}$$

$$\cancel{+ Hp\Theta\frac{\partial}{\partial p}\left[p\frac{\partial f^{(0)}}{\partial p}\right]} - p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]$$

$$\Rightarrow \boxed{\left.\frac{df}{dt}\right|_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\underbrace{\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}}_{\text{free-streaming}} + \underbrace{\frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}}_{\text{gravity}}\right]}$$

first order ~~per~~ Boltzmann equation for photons (still collisionless)

Note: "x" always appears in the combination $\underbrace{\text{"}ax\text{"}}_{\text{physical distance}}$

Collision terms: Compton scattering (MESSY!)

$$e^-(\vec{q}) + \gamma(\vec{p}) \longleftrightarrow e^-(\vec{q}\,') + \gamma(\vec{p}\,')$$

Schematically

$$C[f(\vec{p})] = \sum_{\vec{q},\vec{q}',\vec{p}'} |M|^2 \left\{f_e(\vec{q}\,')f(\vec{p}\,') - f_e(\vec{q})f(\vec{p})\right\}$$

$|M|^2$: amplitude (reversible)

Neglected Bose enhancement and Pauli blocking (fine at 1st order)

+ conservation of momentum : $\vec{p} + \vec{q} = \vec{p}\,' + \vec{q}\,'$

+ conservation of energy: $E(p) + E_e(q) = E(p') + E_e(q')$

Collision term

$$C[f(\bar{p})] = \frac{1}{p}\int\frac{d^3q}{(2\pi)^3 2E_e(q)}\int\frac{d^3q'}{(2\pi)^3 2E_e(q')}\int\frac{d^3p'}{(2\pi)^3 2E(p')}|M|^2(2\pi)^4$$

$$\times\,\delta^3(\bar{p}+\bar{q}-\bar{p}'-\bar{q}')\,\delta\left[E(p)+E_e(q)-E(p')-E_e(q')\right]\left\{f_e(\bar{q}')f(\bar{p}')-f_e(\bar{q})f(\bar{p})\right\}$$

$E(p)=p \quad E(p')=p' \qquad E_e(q)\simeq m_e+\frac{q^2}{2m_e}\ \simeq m_e \qquad E_e(q')\simeq m_e+\frac{q'^2}{2m_e}\ \simeq m_e$

in denominator

Similar to collision term seen earlier, but not integrated over $\bar{p}$ since we are interested in following $f(\bar{p})$

Use δ^3 to do $\bar{q}'$ integral $\longrightarrow \bar{q}'=\bar{p}+\bar{q}-\bar{p}'$

Approximate $E_e(q)\sim m_e \quad E_e(q')\sim m_e$

$$C[f(\bar{p})] \simeq \frac{1}{p}\int\frac{d^3q}{(2\pi)^3 2m_e}\int\frac{d^3p'}{(2\pi)^3 2p'}\int\frac{d^3q'}{(2\pi)^3 2m_e}(2\pi)^4\delta^3(\bar{p}+\bar{q}-\bar{p}'-\bar{q}')\,\cancel{\delta[\ldots]}$$

$$\times\,\delta\left[p+\cancel{m_e}+\frac{q^2}{2m_e}-p'-\cancel{m_e}-\frac{q'^2}{2m_e}\right]|M|^2\left\{f_e(\bar{q}')f(\bar{p}')-f_e(\bar{q})f(\bar{p})\right\}$$

$$=\frac{1}{4m_e^2 p}\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3 p'}\int d^3q'\,\frac{\cancel{(2\pi)^4}\,\pi}{\cancel{2}(2\pi)^3}\,\delta^3(\ldots)\,\delta(\ldots)\,|M|^2\{\ldots\} \ \simeq$$

$\& \ \bar{q}'\Rightarrow\bar{p}+\bar{q}-\bar{p}'$

$$\simeq\frac{\pi}{4m_e^2 p}\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3 p'}\,\delta\left[p+\frac{q^2}{2m_e}-p'-\frac{(\bar{q}+\bar{p}-\bar{p}')^2}{2m_e}\right]|M|^2\left\{f_e(\bar{q}+\bar{p}-\bar{p}')f(\bar{p}')-f_e(\bar{q})f(\bar{p})\right\}$$

Kinematics of non-relativistic Compton scattering: very little energy is transferred!

$$E_e(q)-E_e(\bar{q}+\bar{p}-\bar{p}')=\frac{q^2}{2m_e}-\frac{(\bar{q}+\bar{p}-\bar{p}')^2}{2m_e}\simeq\cancel{\frac{q^2}{2m_e}}-\cancel{\frac{q^2}{2m_e}}-\frac{p^2}{2m_e}-\frac{p'^2}{2m_e}+\frac{(\bar{p}'-\bar{p})\cdot\bar{q}}{m_e}$$

$$\simeq\frac{(\bar{p}'-\bar{p})\cdot\bar{q}}{m_e}\quad\text{as } q\gg p,p'$$

Nearly elastic scattering $p' \simeq p$

so $\bar{p}' - \bar{p} \sim O(p) \sim O(T)$

$$E_e(q) - E_e(\bar{q} + \bar{p} - \bar{p}') \simeq \frac{(\bar{p}' - \bar{p})\cdot\bar{q}}{m_e}$$

$\sim T v_b \quad \left(T\frac{q}{m_e}\right)$

So change in electron energy due to Compton scattering $\sim T v_b$

* Kinetic energy of electron $\sim T \Rightarrow$ Fractional energy change in a single Compton collision small, $O(v_b)$

So expand final electron kinetic energy around zeroth order value

$\frac{(\bar{q}+\bar{p}-\bar{p}')^2}{2m_e}$ $\qquad$ $\frac{q^2}{2m_e}$

Expand δ (tricky?...)

$$\delta\left[p + \frac{q^2}{2m_e} - p' - \frac{(\bar{q}' + \bar{p} - \bar{p}')^2}{2m_e}\right] \simeq \delta(p-p') + \left(E_e(q') - E_e(q)\right) \left.\frac{\partial\delta(p + E_e(q) - p' - E_e(q'))}{\partial E_e(q')}\right|_{E_e(q) = E_e(q')}$$

$$\simeq \delta(p-p') + \frac{(\bar{p}-\bar{p}')\cdot\bar{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'}$$

used $\frac{\partial f(x-y)}{\partial x} = -\frac{\partial f(x-y)}{\partial y}$

$$\rightarrow f_e(\bar{q}+\bar{p}-\bar{p}')f(\bar{p}') - f_e(\bar{q})f(\bar{p}) \simeq f_e(\bar{q})f(\bar{p}') - f_e(\bar{q})f(\bar{p}) = f_e(\bar{q})\left[f(\bar{p}') - f(\bar{p})\right]$$

$f_e(\bar{q}+\bar{p}-\bar{p}') \simeq f_e(\bar{q})$

Putting everything together...

$$C[f(\bar{p})] = \frac{\pi}{4m_e^2 p}\int \frac{d^3q\, f_e(\bar{q})}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3 p'}|M|^2\left\{\delta(p-p') + \frac{(\bar{p}-\bar{p}')\cdot\bar{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'}\right\}\left\{f(\bar{p}') - f(\bar{p})\right\}$$

Later integrate out $\frac{\partial\delta(p-p')}{\partial p'}$ by parts...

$$\int A\frac{\partial\delta}{\partial p'} = A\delta - \int\frac{\partial A}{\partial p'}\delta$$

$|M|^2 \simeq 8\pi \sigma_T^2 m_e$ $\quad$ σ_T: Thomson cross-section

↳ • Ignores angular dependence ~~(Important for polarization)~~

$\lesssim 1\%$ effect in collision term $\quad \propto (1+\cos^2[\hat{p}\cdot\hat{p}'])$

• Ignores polarization $\quad \propto |\hat{\varepsilon}\cdot\hat{\varepsilon}'|^2$

↳ CMB photons are polarized at a very small level

Important, but we ignore it here

So to recap

$$C[f(\bar{p})] \simeq \frac{\pi}{4m_e^2 p}\underbrace{\int \frac{d^3q\, f_e(\bar{q})}{(2\pi)^3}}_{n_e}\int \frac{d^3p'}{(2\pi)^3 p'}\, 8\pi\sigma_T m_e^2 \left\{\delta(p-p') + \frac{(\bar{p}-\bar{p}')\cdot\bar{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'}\right\}\left[f(\bar{p}') - f(\bar{p})\right]$$

Keep only terms first order in energy transfer ($\sim v_b$) like: $\frac{q}{m_e} \sim v_b$

Expand this mess...

$$\int \frac{d^3q\, f_e(\bar{q})}{(2\pi)^3}\frac{\bar{q}}{m_e} \simeq n_e \bar{v}_b$$

$$C[f(\bar{p})] = \frac{2\pi^2\sigma_T n_e}{p}\int \frac{d^3p'}{(2\pi)^3 p'}\left[\delta(p-p') + (\bar{p}-\bar{p}')\cdot\bar{v}_b \frac{\partial\delta(p-p')}{\partial p'}\right]$$

$$f \simeq f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta$$

$$\times\left\{\underbrace{f^{(0)}(\bar{p}') - f^{(0)}(\bar{p})}_{\int \delta(p-p')[f(\bar{p}')-f(\bar{p})]=0}\underbrace{- p'\frac{\partial f^{(0)}}{\partial p'}\Theta(\hat{p}') + p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p})}_{\text{1st order in }\Theta\text{, 2nd order when multiplies } v_b}\right\} \simeq$$

Note: we are expanding simultaneously in $\Theta \ll 1$, $v \ll 1$, neglect Θv !!!

$$\frac{2\pi^2\sigma_T n_e}{(2\pi)^3 p} \qquad \int d^3p' = \int dp'\, p'^2 \int d\Omega'$$

$$\simeq \frac{n_e\sigma_T}{4\pi p}\int_0^\infty dp'\, p' \int d\Omega' \left[\delta(p-p')\left(-p'\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p}') + p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p})\right)\right.$$

$$\left. + (\bar{p}-\bar{p}')\cdot\bar{v}_b \frac{\partial\delta(p-p')}{\partial p'}\left(f^{(0)}(p') - f^{(0)}(p)\right)\right]$$

Only two terms depend on Ω', need to be integrated $\int d\Omega'$

- $\propto \int d\Omega' \, \Theta(\hat{p}')$
- $\propto \int d\Omega' \, \bar{p}' \cdot \bar{v}_b$

Define monopole of the temperature perturbation Θ

$$\Theta_0(\bar{x}, t) \equiv \frac{1}{4\pi} \int d\Omega' \, \Theta(\hat{p}', \bar{x}, t)$$

(momentum) angle-averaged perturbation : CANNOT be absorbed into T since T does not depend on $\bar{x}$ but Θ_0 still does!

Θ_0 at any point: deviation of the monopole at this point from its average across all space

$\int d\Omega' \, \Theta(\hat{p}') \propto \Theta_0$ $\qquad$ $\int d\Omega' \, \bar{p}' \cdot \bar{v}_b = 0$ since $\bar{v}_b$ fixed vector ("odd" function)

$$\Rightarrow C[f(\bar{p})] = \frac{n_e \sigma_T}{4\pi p} \int_0^\infty dp' p' \Big[\delta(p-p') \Big(-4\pi \Theta_0 \, p' \frac{\partial f^{(0)}}{\partial p'} + 4\pi p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}) \Big) + 4\pi \bar{p} \cdot \bar{v}_b \frac{\partial \delta(p-p')}{\partial p'} \Big(f^{(0)}(p') - f^{(0)}(p) \Big) \Big] =$$

$$= \frac{n_e \sigma_T}{p} \int_0^\infty dp\, p' \Big[\underbrace{\delta(p-p') \Big(-p' \frac{\partial f^{(0)}}{\partial p'} \Theta_0 + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}) \Big)}_{\text{trivial: } p' \to p \quad -p^2 \frac{\partial f^{(0)}}{\partial p} (\Theta_0 - \Theta(\hat{p}))} + \underbrace{\bar{p} \cdot \bar{v}_b \frac{\partial \delta(p-p')}{\partial p'} \Big(f^{(0)}(p') - f^{(0)}(p) \Big)}_{\text{By parts}} \Big] =$$

$$\int_0^\infty dp\, p' \frac{\partial \delta(p-p')}{\partial p'} \Big(f^{(0)}(p') - f^{(0)}(p) \Big) = \cancel{\delta(p-p') \times (\dots) \Big|_0^\infty} - \int_0^\infty dp\, p' \, \delta(p-p') \frac{\partial f^{(0)}(p')}{\partial p'} = -p \frac{\partial f^{(0)}}{\partial p}$$

$$= \frac{n_e \sigma_T}{p} \Big[-p^2 \frac{\partial f^{(0)}}{\partial p} (\Theta_0 - \Theta(\hat{p})) - p \frac{\partial f^{(0)}}{\partial p} \, \bar{p} \cdot \bar{v}_b \Big] \overset{\frac{\bar{p}}{p} = \hat{p}}{=} -p \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T \Big[\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \bar{v}_b \Big]$$

Final result for collision term

$$\boxed{C[f(\vec{p})] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right]}$$

Physical interpretation:

if $\vec{v}_b \to 0$ (electrons no bulk velocity)

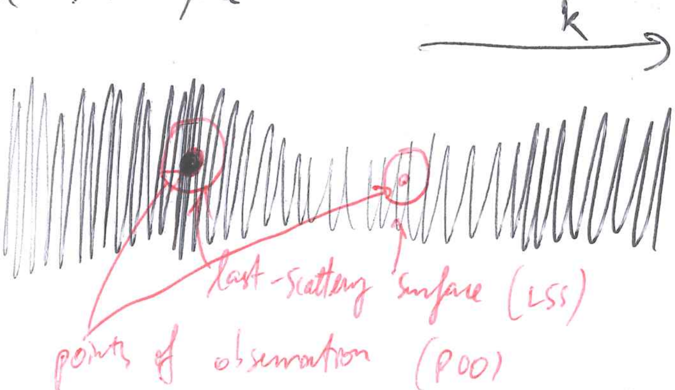
$$\Theta(\hat{p}) \to \Theta_0$$

↳ efficient Compton scattering makes only monopole of perturbation survive because of small mean-free-path

At a given point in space photons from all directions have roughly the same temperature (uniform on sky)

↳ monopole varies across space!

Pictorial example



$T_{LSS} \sim T_{POO}$ throught the whole of space

If electrons also have bulk velocity, photon temperature perturbation will have dipole moment (but higher order moments suppressed)

Higher order ~~moments~~ multipole moments / Legendre polynomial expansion

$$\mu \equiv \frac{\vec{k}\cdot\hat{p}}{k} = \cos\theta \qquad \Theta_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)$$

$P_0(\mu) = 1$ $\quad P_2(\mu) = \frac{3\mu^2-1}{2}$

$P_1(\mu) = \mu$ $\quad P_3(\mu) = \frac{5\mu^3-3\mu}{2}$

$$\Theta(\mu) = \sum_{\ell=0}^{\infty}\Theta_\ell P_\ell(\mu)$$

Properties: $\int_{-1}^{1} d\mu\, P_m(\mu) P_n(\mu) = 0$ if $n \neq m$

$\frac{2}{2n+1}\delta_{mn}$

$$\frac{1}{\sqrt{1-2\mu t + t^2}} = \sum_{n=0}^{\infty} P_n(\mu) t^n$$