Now let's collect zeroth and first order terms

$\boxed{\text{0th order}}$

$$\frac{\partial n_{dm}}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}\overset{\text{1st order}}{\cancel{(n_{dm}v^i)}} + 3\left[\frac{da/dt}{a} + \overset{\text{1st order}}{\cancel{\frac{\partial\Phi}{\partial t}}}\right]n_{dm} = 0$$

$$\hookrightarrow \quad \underbrace{\frac{\partial n_{dm}^{(0)}}{\partial t} + 3\,\frac{da/dt}{a}\,n_{dm}^{(0)} = 0}$$

$$= \frac{1}{a^3}\frac{d}{dt}\left(n_{dm}^{(0)}a^3\right) = \frac{1}{\cancel{a^3}}\cancel{a^3}\frac{\partial n_{dm}^{(0)}}{\partial t} + \frac{3\cancel{a^2}\dot{a}}{a^{3}}n_{dm}^{(0)} = \frac{\partial n_{dm}^{(0)}}{\partial t} + 3Hn_{dm}^{(0)}$$

$$\frac{d}{dt}\left(n_{dm}^{(0)}a^3\right) = 0 \Longrightarrow \boxed{n_{dm}^{(0)} \propto a^{-3}}$$

confirms that $\rho_{dm} \propto a^{-3}$ as expected for dust component

$\boxed{\text{1st order}}$ (terms $\sim n_{dm}v$, $\sim n_{dm}\Phi \longrightarrow$ set $n_{dm} = n_{dm}^{(0)}$)

$\hookrightarrow$ "small" quantities: Φ, δ, v

$$n_{dm} = n_{dm}^{(0)}\left[1 + \delta(\bar{x}, t)\right] \qquad \text{so} \quad n_{dm}^{(1)} = n_{dm}^{(0)}\delta$$

δ

δ: fractional overdensity $\quad \frac{\delta\rho_{dm}}{\rho_{dm}}$

$$\frac{\partial}{\partial t}\Big[\overset{\text{0th}}{\cancel{n_{dm}^{(0)}}} \underbrace{+ n_{dm}^{(0)}\delta}_{\text{1st}}\Big] + \frac{1}{a}\frac{\partial}{\partial x^i}\Big[\underbrace{\cancel{n_{dm}^{(0)}}v^i}_{\text{1st}} \overset{\text{2nd}}{\cancel{+ n_{dm}^{(0)}\delta v_i}}\Big] + 3\left[\frac{da/dt}{a} + \frac{\partial\Phi}{\partial t}\right]\left(n_{dm}^{(0)} + n_{dm}^{(0)}\delta\right)$$

$$\simeq \frac{\partial}{\partial t}\left[n_{dm}^{(0)}\delta\right] + \frac{1}{a}\frac{\partial}{\partial x^i}\left[n_{dm}^{(0)}v^i\right] + 3H\left(\overset{\text{0th}}{\cancel{n_{dm}^{(0)}}} + n_{dm}^{(0)}\delta\right) + 3\frac{\partial\Phi}{\partial t}\left(n_{dm}^{(0)} \overset{\text{2nd}}{\cancel{+ n_{dm}^{(0)}\delta}}\right)$$

$$\simeq \frac{\partial n_{dm}^{(0)}}{\partial t}\delta + \frac{\partial\delta}{\partial t}n_{dm}^{(0)} + \frac{1}{a}\frac{\partial n_{dm}^{(0)}}{\partial x^i}v^i + \frac{1}{a}n_{dm}^{(0)}\frac{\partial v^i}{\partial x^i} + \cancel{3Hn_{dm}^{(0)}} + 3Hn_{dm}^{(0)}\delta + 3\frac{\partial\Phi}{\partial t}n_{dm}^{(0)}$$

$$\frac{\partial}{\partial t}\left[n_{dm}^{(0)}\right] = \frac{\partial}{\partial t}\left[n_{dm,0}\,a^{-3}\right] = n_{dm,0} \times -3\frac{a^{4}}{\dot{a}} = -3n_{dm,0}a^{-3}\frac{\dot{a}}{a} = -3H\underbrace{n_{dm,0}\,a^{-3}}_{n_{dm}^{(0)}} = -3Hn_{dm}^{(0)}$$

$$= \cancel{-3Hn_{dm}^{(0)}\delta} + n_{dm}^{(0)}\frac{\partial\delta}{\partial t} + \cancel{\frac{v^i}{a}\frac{\partial n_{dm}^{(0)}}{\partial x^i}} + \frac{1}{a}n_{dm}^{(0)}\frac{\partial v^i}{\partial x^i} + \cancel{3Hn_{dm}^{(0)}\delta} + 3\frac{\partial\Phi}{\partial t}n_{dm}^{(0)} =$$

$$= \cancel{n_{dm}^{(0)}}\frac{\partial \delta}{\partial t} + \frac{1}{a}\cancel{n_{dm}^{(0)}}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t}\cancel{n_{dm}^{(0)}} = 0$$

$$\Rightarrow \boxed{\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0}$$

Boltzmann equation for cold DM overdensity ~~(first order / 0th moment)~~ ~~1st~~ (1st order / 0th moment)

Two variables: δ, v^i

δ ↑ 0th moment

v^i ← 1st moment

Need another equation

Return to unintegrated Boltzmann equation, but this time take the 1st instead of 0th $\left(\int \times \frac{d^3p}{(2\pi)^3}\right)$ moment

$$\int \times \frac{d^3p}{(2\pi)^3}\frac{p}{E}\hat{p}^j \quad \frac{p}{E}\hat{p}^j \to \hat{v}^j$$

$$0 = \int \frac{d^3p}{(2\pi)^3}\frac{p\hat{p}^j}{E}\frac{\partial f_{dm}}{\partial t} + \int \frac{d^3p}{(2\pi)^3}\frac{p\hat{p}^j}{E}\frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{dm}}{\partial x^i} - \int \frac{d^3p}{(2\pi)^2}\frac{p\hat{p}^j}{E}\left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t}\right]\frac{p^2}{E}\frac{\partial f_{dm}}{\partial E}$$

$$- \int \frac{d^3p}{(2\pi)^3}\frac{p\hat{p}^j}{E}\frac{\hat{p}^i p}{a}\frac{\partial \Psi}{\partial x^i}\frac{\partial f_{dm}}{\partial E}$$

$$\underbrace{\frac{\partial}{\partial t}\int \frac{d^3p}{(2\pi)^3} f_{dm}\frac{p\hat{p}^j}{E}}_{\frac{\partial}{\partial t}\left[n_{dm}\hat{v}_j\right]} + \cancel{\frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{d^3p}{(2\pi)^3} f_{dm}\frac{p^2\hat{p}^i\hat{p}^j}{E^2}} - \left[H + \frac{\partial \Phi}{\partial t}\right]\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{dm}}{\partial E}\frac{p^3\hat{p}^j}{E^2} - \frac{1}{a}\frac{\partial \Psi}{\partial x^i}\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{dm}}{\partial E}\frac{p^2\hat{p}^i\hat{p}^j}{E}$$

$\mathcal{O}\left(\left(\frac{p}{E}\right)^2\right) \sim v^2$

$$\int \frac{d^3p}{(2\pi)^3}\frac{\partial f_{dm}}{\partial E}\frac{p^3\hat{p}^j}{E^2} = \int \frac{d^3p}{(2\pi)^3}\frac{E}{p}\frac{p^3\hat{p}^j}{E^2}\frac{\partial f_{dm}}{\partial p} = \int \frac{d^3p}{(2\pi)^3}\frac{p^2\hat{p}^j}{E}\frac{\partial f_{dm}}{\partial p} =$$

$$\left\{\frac{dE}{dp} = \frac{p}{E} \rightarrow \frac{\partial}{\partial E} = \frac{E}{p}\frac{\partial}{\partial p}\right\} = \int \frac{d\Omega\,\hat{p}^j}{(2\pi)^3}\int dp\,\frac{p^4}{E}\frac{\partial f_{dm}}{\partial p} =$$

↖ by parts

$$= - \cancel{\int \frac{d\Omega \, \hat{p}^j}{(2\pi)^3} f_{dm} \frac{\partial}{\partial p}\left(\frac{p^4}{E}\right)}$$

$$\frac{\partial}{\partial p}\left(\frac{p^4}{E}\right) = \frac{4p^3 E - p^4 \frac{dE}{dp}}{E^2} = \frac{4p^3 E - p^4 \frac{p}{E}}{E^2} = \frac{4p^3 E - \frac{p^5}{E}}{E^2} = 4\frac{p^3}{E} - \frac{p^5}{E^3}$$

$$= - \int \frac{d\Omega \, \hat{p}^j}{(2\pi)^3} \int dp \, f_{dm} \frac{\partial}{\partial p}\left(\frac{p^4}{E}\right) = - \int \frac{d\Omega \, \hat{p}^j}{(2\pi)^3} \int_0^\infty dp \, f_{dm} \left(\frac{4p^3}{E} - \cancel{\frac{p^5}{E^3}} \right) \xrightarrow{} O\left(\left(\frac{p}{E}\right)^3\right) \sim v^3 !!!$$

$$\simeq -4 \int \frac{d\Omega \, \hat{p}^j}{(2\pi)^3} \int_0^\infty dp \, \frac{p^3}{E} f_{dm} = -4 \underbrace{\int \frac{d\Omega}{(2\pi)^3} \int_0^\infty dp \, p^2}_{\int \frac{d^3p}{(2\pi)^3}} \frac{p \, \hat{p}^j}{E} f_{dm} = -4 \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{p \, \hat{p}^j}{E} f_{dm}}_{n_{dm} v^j} =$$

$$= -4 n_{dm} v_j$$

$$\rightarrow \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial E} \frac{\hat{p}^i \hat{p}^j p^2}{E} = \int \frac{d^3p}{(2\pi)^3} \frac{E}{p} \frac{\partial f_{dm}}{\partial p} \frac{\hat{p}^i \hat{p}^j p^2}{E} = \int \frac{d^3p}{(2\pi)^3} \hat{p}^i \hat{p}^j p \frac{\partial f_{dm}}{\partial p} =$$

$$= \underbrace{\int d\Omega \, \hat{p}^i \hat{p}^j}_{\delta^{ij} \frac{4\pi}{3}} \int \frac{dp}{(2\pi)^3} p^3 \frac{\partial f_{dm}}{\partial p} = -\delta^{ij} \frac{4\pi}{3} \int \frac{dp}{8\pi^3} 3p^2 f_{dm} = -\delta^{ij} \frac{4\pi \, 3}{3 \times 8\pi^3 \, 2} \int dp \, p^2 f_{dm} =$$

$$= - \boxed{\frac{\delta^{ij}}{2\pi^2} \int dp \, p^2 f_{dm}} \longrightarrow \underbrace{\int \frac{d^3p}{(2\pi)^3} f_{dm}}_{n_{dm}} = \frac{4\pi}{8\pi^3} \int dp \, p^2 f_{dm} = \frac{1}{2\pi^2} \int dp \, p^2 f_{dm}$$

$$= - \delta^{ij} n_{dm}$$

$$\rightarrow -\frac{1}{a} \frac{\partial \Psi}{\partial x^i} \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial E} \frac{p^2 \hat{p}^i \hat{p}^j}{E} = -\frac{1}{a} \frac{\partial \Psi}{\partial x^i} \times - \delta^{ij} n_{dm} = \frac{n_{dm}}{a} \frac{\partial \Psi}{\partial x^j}$$

Putting everything together, 1st moment of Boltzmann equation:

2nd order

$$\frac{\partial}{\partial t}(n_{dm} v^j) - 4 n_{dm} v_j \times \left[\frac{da/dt}{a} + \cancel{\frac{\partial \Phi}{\partial t}} \right] + \frac{n_{dm}}{a} \frac{\partial \Psi}{\partial x^j} \simeq \frac{\partial}{\partial t}(n_{dm} v_j) + 4 H n_{dm} v_j + \frac{n_{dm}}{a} \frac{\partial \Psi}{\partial x^j} = 0$$

v, Ψ 1st order, so there are no 0th order terms

$\hookrightarrow$ can set $n_{dm} = n_{dm}^{(0)}$ everywhere (otherwise we get 2nd order terms)

$$\frac{\partial}{\partial t}\left(n_{dm} v^j\right) + 4H\, n_{dm} v^j + \frac{n_{dm}}{a}\frac{\partial \psi}{\partial x^j} = 0$$

$$\downarrow$$

$$\frac{\partial}{\partial t}\left(n_{dm}^{(0)} v^j\right) + 4H n_{dm}^{(0)} v^j + \frac{n_{dm}^{(0)}}{a}\frac{\partial \psi}{\partial x^j} = 0$$

$$\downarrow \quad n_{dm}^{(0)} \propto a^{-3} \rightarrow \frac{\partial}{\partial t} n_{dm}^{(0)} = -3H n_{dm}^{(0)}$$

$$\cancel{-3H \cancel{n_{dm}^{(0)}} v^j} + \cancel{n_{dm}^{(0)}}\frac{\partial v^j}{\partial t} + \cancel{4}H\cancel{n_{dm}^{(0)}} v^j + \frac{\cancel{n_{dm}^{(0)}}}{a}\frac{\partial \psi}{\partial x^j} = 0$$

$$\Downarrow$$

$$\boxed{\frac{\partial v^j}{\partial t} + H v^j + \frac{1}{a}\frac{\partial \psi}{\partial x^j} = 0} \; *$$

* Boltzmann equation for cold DM velocity (1st order / 1st moment)

↳ no $(\vec{v}\cdot\vec{\nabla})\vec{v}$ term since it is 2nd order!

Note: the equation for the nth moment depends on the (N+1)th moment → infinite hierarchy (in principle)

$\delta \rightarrow v$

$v \rightarrow$ quadrupole/anisotropic stress? → Dropped terms $O\left(\left(\frac{p}{E}\right)^2\right)$!!! Because DM is cold

↳ sufficient to characterise CDM!!!

↳ Generic feature of integrating Boltzmann equation to get fluid equations

~~*Conformal~~ time

Equations for δ and v

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0 \qquad \xRightarrow[\quad]{\substack{\text{conformal time} \\ d\eta = \frac{dt}{a} \,\rightarrow\, \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta} = \frac{1}{a}\,\dot{}}} \qquad \dot{\delta} + \frac{\partial v^i}{\partial x^i} + 3\dot{\Phi} = 0$$

$$\frac{\partial v^j}{\partial t} + H v^j + \frac{1}{a}\frac{\partial \psi}{\partial x^j} = 0 \qquad\qquad \dot{v}_j + \underbrace{\frac{\dot{a}}{a} v_j}_{\mathcal{H} v_j} + \frac{\partial \psi}{\partial x^j} = 0$$

As usual take Fourier transform, assume irrotational velocity ($\bar{v} \propto \hat{k} \longrightarrow \tilde{v}^i = \left(\frac{k^i}{k}\right)\tilde{v} \longrightarrow \left(\frac{\partial \tilde{v}}{\partial x^i}\right) = ik\tilde{v}$)

$$\longrightarrow \boxed{\begin{aligned} &\dot{\tilde{\delta}} + ik\tilde{v} + 3\dot{\tilde{\Phi}} = 0 \\ &\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0 \end{aligned}}$$

Collisionless Boltzmann equation for DM:
- to 1st order
- in Fourier space
- expressed in terms of 2 lowest order moments δ, v (since DM is cold!!!)

Boltzmann equations for baryons

Baryons: $\underbrace{\text{electrons + protons}}$ (bad nomenclature)

coupled by Coulomb scattering $e + p \to e + p$

rate always larger than H

$$\longrightarrow \frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}} \equiv \delta_b$$

strong scattering rate forces common overdensity

$$\bar{v}_e = \bar{v}_p \equiv \bar{v}_b$$

similarly for velocity

So similarly to dark matter we want the equations for δ_b, v_b

Start from unintegrated equations for e, p:

$$\frac{df_e}{dt}(\bar{x}, \bar{q}, t) = \underbrace{\langle C_{ep} \rangle_{QQ'q'}} + \langle C_{e\gamma} \rangle_{pp'q'} \longrightarrow \text{Compton scattering}$$

$$e^-(\bar{q}) + \gamma(\bar{p}) \to e^-(\bar{q}') + \gamma(\bar{p}')$$

$$\frac{df_p}{dt}(\bar{x}, \bar{Q}, t) = \langle C_{ep} \rangle_{qq'Q'}$$

Coulomb scattering

$$e^-(\bar{q}) + p^+(\bar{Q}) \longrightarrow e^-(\bar{q}') + p^+(\bar{Q}')$$

Compact notation (compare Boltzmann equations for photons)

Unintegrated part of collision term $\rightarrow 2^3$

$$C_{e\gamma} \equiv (2\pi)^4 \delta^4(p+q-p'-q') \frac{|M|^2_{\text{Compton}}}{\boxed{8}\, E_\gamma(p) E_\gamma(p') E_e(q') E_e(q)} \left[f_e(q') f_\gamma(p') - f_e(q) f_\gamma(p) \right]$$

$$\langle (\ldots) \rangle_{pp'q'} \equiv \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} (\ldots)$$

Similarly for C_{ep} : $|M|^2_{\text{Compton}} \rightarrow |M|^2_{\text{Coulomb}}$ and integrated momenta are different

$\langle C_{p\gamma} \rangle$ neglected since $\sigma \propto \frac{1}{m_p^2}$, so interactions of combined electron-proton fluid with photons driven by Compton scattering between electrons and photons

(Also neglecting ionization and recombination terms)

Now multiply $\frac{df_e}{dt}$ equation by $\frac{d^3q}{(2\pi)^3}$ and integrate $\rightarrow$ LHS identical to DM, recall:

$$\frac{df_e}{dt} = \frac{\partial f_e}{\partial t} + \frac{\partial f_e}{\partial x^i} \frac{\hat{p}^i}{a} \frac{p}{E} - \frac{\partial f_e}{\partial E} \left[H \frac{p^2}{E} + \frac{p^2}{E} \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i p}{a} \frac{\partial \Psi}{\partial x^i} \right] = 0$$

$$\Bigg\downarrow \int \times \frac{d^3q}{(2\pi)^3}$$

$$\cancel{\frac{\partial n_e}{\partial t} + \frac{1}{a} \frac{\partial (n_e v_b^i)}{\partial x^i} + 3 \left[H + \frac{\partial \Phi}{\partial t} \right] n_e = \langle C_{ep} \rangle_{QQ'q'q} + \langle C_{e\gamma} \rangle_{pp'q'q}}$$

$$\frac{\partial n_e}{\partial t} + \frac{1}{a} \frac{\partial (n_e v_b^i)}{\partial x^i} + 3 \left[H + \frac{\partial \Phi}{\partial t} \right] n_e = \cancelto{0}{\langle C_{ep} \rangle_{QQ'q'q}} + \cancelto{0}{\langle C_{e\gamma} \rangle_{pp'q'q}}$$

$\equiv 0$ mathematical physical identity, not approximation why?, Take e.g.

$$\langle C_{ep} \rangle_{QQ'q'q}$$

$\langle \ldots \rangle_{QQ'q'q}$ symmetric in $Q \Leftrightarrow Q'$ $q \Leftrightarrow q'$

C_{ep} antisymmetric in $Q \Leftrightarrow Q'$ $q \Leftrightarrow q'$ because of

~~$f_e(q')f_p(Q') - f_e(q)$~~ $f_e(q)f_p(Q) - f_e(q')f_e(Q')$

So integral is identically 0!

Physically speaking, Coulomb scattering conserves electron number, so <u>cannot</u> contribute to $\frac{\partial n_e}{\partial t}$

$\int f_e(q')f_p(Q')$ $\propto$ # electrons produced in Coulomb scattering

$=$

$\int f_e(q)f_p(Q)$ $\propto$ # electrons lost in Coulomb scattering

Similarly for Compton scattering

True in general: multiplying unintegrated collision term by conserved quantity and integrating $\longrightarrow 0$

So for δ_b we have the exact same equation as cold DM.

Using conformal time and going to Fourier space:

$$\boxed{\dot{\tilde{\delta}}_b + ik\tilde{v}_b + 3\dot{\tilde{\Phi}} = 0}$$

Boltzmann equation for baryon overdensity (1st order/0th moment)

As with DM we also need an equation for the velocity, take first moments of $\frac{df_e}{dt}$ and $\frac{df_p}{dt}$ equations

~~$\int \times \frac{d^3q}{(2\pi)^3}\frac{q}{E}$~~ ~~$\int \times \frac{d^3Q}{(2\pi)^3}\frac{Q}{E}$~~

$$\int \frac{d^3q}{(2\pi)^3}\,\bar{q}\,\frac{df_e}{dt}(\bar{x},\bar{q},t) = \langle C_{ep}\,\bar{q}\rangle_{QQ'q'q} + \langle C_{e\gamma}\,\bar{q}\rangle_{pp'q'q}$$

$$\int \frac{d^3\bar{Q}}{(2\pi)^3}\,\bar{Q}\,\frac{df_p}{dt}(\bar{x},\bar{q},t) = \langle C_{ep}\,\bar{Q}\rangle_{\cancel{QQ'q}\; qq'Q'Q}$$

Recall for DM we did the same but with $\int \frac{d^3p}{(2\pi)^3}\,\frac{p\hat{p}^i}{E} = \int \frac{d^3p}{(2\pi)^3}\,\frac{\bar{p}}{E}$

So here we can borrow the same results up to a factor of $E \approx m_e, m_p$

For DM we got

$$\frac{\partial(n_{dm} v^J)}{\partial t} + 4H\,n_{dm} v^J + \frac{n_{dm}}{a}\frac{\partial\Psi}{\partial x^J} = 0$$

Here we get:

$$m_e\frac{\partial(n_b v_b^J)}{\partial t} + 4H\,n_b v_b^J m_e + \frac{n_b}{a}m_e\frac{\partial\Psi}{\partial x^J} = \langle C_{ep}\,\bar{q}\rangle_{QQ'q'q} + \langle C_{e\gamma}\,\bar{q}\rangle_{pp'q'q}$$

$$+\;m_p\frac{\partial(n_b v_b^J)}{\partial t} + 4H\,m_p n_b v_b^J + m_p\frac{n_b}{a}\frac{\partial\Psi}{\partial x^J} = \langle C_{ep}\,\bar{Q}\rangle_{QQ'q'q}$$

$$(\cancel{m_e}+m_p)\left[\frac{\partial(n_b v^J)}{\partial t} + 4H\,n_b v_b^J + \frac{n_b}{a}\frac{\partial\Psi}{\partial x^J}\right] = \underbrace{\cancel{\langle C_{ep}(q^J+Q^J)\rangle_{QQ'q'q}}}_{0} + \langle C_{e\gamma}\,q^J\rangle_{pp'q'q}$$

$m_p \gg m_e$

0 again mathematically since q^J+Q^J symmetric AND conserved!!!

otherwise get 2nd order terms

$$n_b \longrightarrow n_b^{(0)} \propto a^{-3} \quad \text{so} \quad \frac{\partial}{\partial t}n_b^{(0)} = -3Hn_b^{(0)}$$

$$m_p\frac{\partial}{\partial t}n_b^{(0)} v^J + m_p n_b^{(0)}\frac{\partial v^J}{\partial t} + 4H m_p n_b^{(0)} v_b^J + m_p\frac{n_b^{(0)}}{a}\frac{\partial\Psi}{\partial x^J} = \langle C_{e\gamma}\,q^J\rangle_{pp'q'q}$$

$$\cancel{-3Hm_p n_b^{(0)} v^J} + m_p n_b^{(0)}\frac{\partial v^J}{\partial t} + \cancel{4}H m_p n_b^{(0)} v_b^J + m_p\frac{n_b^{(0)}}{a}\frac{\partial\Psi}{\partial x^J} = \langle C_{e\gamma}\,q^J\rangle_{pp'q'q}$$

$$\longrightarrow \frac{m_p n_b^{(0)}}{\rho_b}\frac{\partial v_b^J}{\partial t} + \frac{H m_p n_b^{(0)} v_b^J}{\rho_b} + \frac{m_p n_b^{(0)}}{a\,\rho_b}\frac{\partial \Psi}{\partial x^J} = \frac{\langle C_{e\gamma} q^J \rangle_{pp'q'q}}{\rho_b}$$

$$\Big\downarrow \; \rho_b \simeq m_p n_b^{(0)}$$

$$\frac{\partial v_b^J}{\partial t} + H v_b^J + \frac{1}{a}\frac{\partial \Psi}{\partial x^J} = \frac{1}{\rho_b}\boxed{\langle C_{e\gamma} q^J \rangle_{pp'q'q}}$$

$\langle C_{e\gamma}\vec{q}\rangle_{pp'q'q} = -\langle C_{e\gamma}\vec{p}\rangle_{pp'q'q}$ since $\langle C_{e\gamma}(\vec{q}+\vec{p})\rangle_{pp'q'q} = 0$

by the conservation of momentum

$-\frac{\langle C_{e\gamma}\vec{p}\rangle_{pp'q'q}}{\rho_b}$ → go to Fourier space

→ multiply by $\hat{k}_J$ (similarly for LHS)

$\hat{k}\cdot\vec{p} = p\mu$ →

$$-\frac{\langle C_{e\gamma}\,p\mu\rangle_{pp'q'q}}{\rho_b}$$

Recall we already computed $\langle C_{e\gamma}\rangle_{p'q'q}$ for the photon Boltzmann equation

$$\langle C_{e\gamma}\rangle_{p'q'q} = -p\frac{\partial f^{(0)}}{\partial p} n_e\sigma_T\left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right]$$

$$-\frac{\langle C_{e\gamma}\,p\mu\rangle_{pp'q'q}}{\rho_b} \xleftarrow{\text{Fourier space}} = \frac{n_e\sigma_T}{\rho_b}\int\frac{d^3p}{(2\pi)^3}\,p^2\frac{\partial f^{(0)}}{\partial p}\mu\left[\tilde{\Theta}_0 - \tilde{\Theta}(\mu) + \cancel{\text{[illegible]}} + \tilde{v}_b\mu\right] =$$

$$= \frac{n_e\sigma_T}{\rho_b}\int_0^\infty \frac{dp}{2\pi^2}\,p^4\frac{\partial f^{(0)}}{\partial p}\int_{-1}^{1}\frac{d\mu}{2}\,\mu\left[\tilde{\Theta}_0 - \tilde{\Theta}(\mu) + \tilde{v}_b\mu\right]$$

~~$\frac{4\pi}{(2\pi)}$~~ ←

$$\int\frac{d^3p}{(2\pi)^3} = \int\frac{dp\,p^2}{(2\pi)^3}\int d\varphi\int d(\cos\theta) = \int\frac{dp\,p^2}{8\pi^3}2\pi\int d\mu =$$

$$= \int\frac{dp\,p^2}{4\pi^2}\int d\mu = \int\frac{dp\,p^2}{2\pi^2}\int\frac{d\mu}{2}$$

$$\frac{n_e\sigma_T}{\rho_b}\underbrace{\int_0^\infty \frac{dp}{2\pi^2}\,p^4\frac{\partial f^{(0)}}{\partial p}}\boxed{\int_{-1}^{1}\frac{d\mu}{2}\,\mu\left[\hat{\Theta}_0-\tilde{\Theta}(\mu)+\tilde{v}_b\mu\right]}$$

$$\Rightarrow \int_0^\infty \frac{dp}{2\pi^2}p^4\frac{\partial f^{(0)}}{\partial p} = -4\int_0^\infty \frac{dp}{2\pi^2}p^3 f^{(0)} = -4\int_0^\infty \frac{dp\,p^2}{2\pi^2}\,p\,f^{(0)} = -4\int\frac{d^3p}{(2\pi)^3}\,p\,f^{(0)} = -4\rho_\gamma$$

$$\int_{-1}^{1}\frac{d\mu}{2}\,\mu\left[\underbrace{\cancel{\tilde{\Theta}_0}}_{0\text{ (odd)}}-\tilde{\Theta}(\mu)+\underbrace{\tilde{v}_b\mu}\right] = i\tilde{\Theta}_1+\frac{\tilde{v}_b}{3}$$

$$\int_{-1}^{1}\frac{d\mu}{2}\tilde{v}_b\mu^2 = \tilde{v}_b\left.\frac{\mu^3}{6}\right|_{-1}^{1} = \tilde{v}_b\left(\frac{1}{6}+\frac{1}{6}\right) = \frac{\tilde{v}_b}{3}$$

Recall $\quad \Theta_l \equiv \frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\Theta(\mu) \qquad P_1(\mu)=\mu$

so $\quad i\int_{-1}^{1}\frac{d\mu}{2}\mu\tilde{\Theta}(\mu) = \tilde{\Theta}_1 \rightarrow -\int\frac{d\mu}{2}\mu\tilde{\Theta}(\mu) = i\tilde{\Theta}_1$

$$\longrightarrow = -4\frac{n_e\sigma_T}{\rho_b}\rho_\gamma\left(i\tilde{\Theta}_1+\frac{\tilde{v}_b}{3}\right)$$

~~Putting everything together~~

~~$\frac{\partial v_b^j}{\partial t}+\frac{da/dt}{a}v_b^j+\frac{1}{a}\frac{\partial\psi}{\partial x^j} = -\frac{n_e\sigma_T}{\rho_b}\rho_\gamma\left(i\Theta_1+\right.$~~

Putting everything together

LHS: $\quad \frac{\partial v_b^j}{\partial t}+\frac{da/dt}{a}v_b^j+\frac{1}{a}\frac{\partial\psi}{\partial x^j} \xrightarrow{t\to\eta} \frac{1}{a}\dot{v}_b+\frac{\dot{a}}{a^2}v_b^j+\frac{1}{a}\frac{\partial\psi}{\partial x^j} \xrightarrow{FT} \frac{1}{a}\dot{\tilde{v}}_b+\frac{\dot{a}}{a^2}\tilde{v}_b+\frac{ik\tilde{\psi}}{a}$

RHS: $\quad -4n_e\sigma_T\frac{\rho_\gamma}{\rho_b}\left(i\tilde{\Theta}_1+\frac{\tilde{v}_b}{3}\right)$ already in Fourier space!

$$\longrightarrow \frac{1}{a}\dot{\tilde{v}}_b+\underbrace{\frac{\dot{a}}{a^2}}_{\mathcal{H}}\tilde{v}_b+\frac{ik\tilde{\psi}}{a} = -4n_e\sigma_T\frac{\rho_\gamma}{\rho_b}\left(i\tilde{\Theta}_1+\frac{\tilde{v}_b}{3}\right)$$

$$\Rightarrow \dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + ik\tilde{\Psi} = \boxed{-4n_e\sigma_T a}\,\frac{\rho_\gamma}{\rho_b}\left(i\tilde{\Theta}_1 + \frac{\tilde{v}_b}{3}\right)$$

Recall definition of optical depth

$$\tau(\eta) \equiv \int_\eta^{\eta_0} d\eta' n_e\sigma_T a \longrightarrow \dot{\tau} \equiv \frac{d\tau}{d\eta} = -n_e\sigma_T a$$

$$\boxed{\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + ik\tilde{\Psi} = \dot{\tau}\frac{4\rho_\gamma}{3\rho_b}\left[3i\tilde{\Theta}_1 + \tilde{v}_b\right]} \quad *$$

Boltzmann equation for baryon velocity (1st order/1st moment)

$\tilde{\Theta}_1$ "$\sim$" v_γ so $[\ldots] \sim v_\gamma - v_b$: momentum exchange

Physical meaning of $\frac{1}{\rho_b}$ factor : moving e^- is difficult because they are tightly coupled to heavy protons. If $m_p \to \infty$, $\rho_b \to \infty$, Compton scattering cannot change electron velocity

$$R \equiv \frac{3\rho_b}{4\rho_\gamma} = \frac{(1+w_b)\rho_b}{(1+w_\gamma)\rho_\gamma}$$ ~~ph~~ baryon-to-photon density ratio

* even if there is neutral hydrogen so $n_e \neq n_b$, approximately valid

Recap Boltzmann equations so far

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau}\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b\right]$$

$$\dot{\tilde{\delta}}_{dm} + ik\tilde{v}_{dm} = -3\dot{\tilde{\Phi}}$$

$$\dot{\tilde{v}}_{dm} + \frac{\dot{a}}{a}\tilde{v}_{dm} = -ik\tilde{\Psi}$$

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_b = -3\dot{\tilde{\Phi}}$$

$$\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b = -ik\tilde{\Psi} + \frac{\dot{\tau}}{R}\left[\tilde{v}_b + 3i\tilde{\Theta}_1\right]$$