First multipole moments

$$\Theta_\ell = \frac{1}{(-i)^\ell} \int_{-1}^{1} \frac{d\mu}{2} \mathcal{P}_\ell(\mu)\, \Theta(\mu)$$

$\ell=2$ quadrupole

$$\Theta_2 = -\int_{-1}^{1} d\mu \frac{(3\mu^2\Theta - \Theta)}{4} = \int_{-1}^{1} \frac{d\mu}{4}(\Theta - 3\mu^2\Theta)$$

$\ell=0$ monopole

$$\Theta_0 = \int_{-1}^{1} \frac{d\mu}{2} \Theta(\mu) = \frac{1}{4\pi}\int d\Omega' \Theta(\vec{x}, \hat{p}', t)$$

$$\int d\Omega' = \int d(\cos\theta)\, d\varphi = 2\pi \int d\mu$$

$\ell=1$ dipole

$$\Theta_1 = i\int_{-1}^{1} \frac{d\mu}{2} \mu\, \Theta(\mu)$$

Compton scattering produces Θ with basically only non-vanishing monopole Θ_0 and dipole Θ_1 (if electrons have a non-zero bulk velocity)

$\underbrace{}_{\text{photons behave like a fluid}}$: tightly coupled photon-electron fluid

Now let's add collision term to the Boltzmann equations

Previously we saw

$$\left.\frac{df}{dt}\right|_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]$$

$$C[f] = -p\frac{\partial f^{(0)}}{\partial p} n_e\sigma_T\left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right]$$

$$\frac{df}{dt} = C[f] \Longrightarrow \boxed{\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T\left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right]}$$

More convenient to introduce conformal time η $\quad d\eta = \frac{dt}{a} \quad \frac{d}{d\eta} = a\frac{d}{dt}$

$$\dot{} = \frac{d}{d\eta} = a\frac{d}{dt} \quad \text{or} \quad \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta} = \frac{\dot{}}{a} \quad \text{so} \quad \frac{\partial\Theta}{\partial t} = \frac{1}{a}\dot{\Theta} \quad \frac{\partial\Phi}{\partial t} = \frac{\dot{\Phi}}{a}$$

$$\frac{\dot{\Theta}}{a} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\dot{\Phi}}{a} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T\left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right]$$

$$\Rightarrow \boxed{\dot{\Theta} + \hat{p}^i \frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \hat{p}^i \frac{\partial \Psi}{\partial x^i} = n_e \sigma_T a \left[\Theta_0 - \Theta + \hat{p} \cdot \vec{v}_b\right]}$$

Partial differential linear equation in η, x

Move to Fourier space!

PDE $\longrightarrow$ ODE for each Fourier mode

Linear: Fourier modes evolve independently

Mode $\bar{k}_x$ evolves without caring about k_y, can solve for each mode independently

$\longrightarrow$ FT gives a set of uncoupled ODEs for Fourier modes

Linearity: ~valid for perturbations to CMB, ~~or~~ and large-scale perturbations at late times

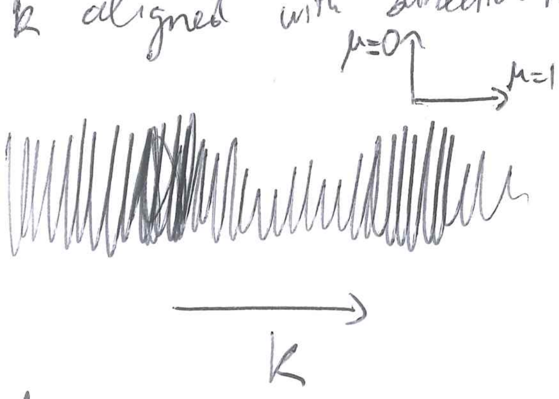
$$\Theta(\bar{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\bar{k}\cdot\bar{x}}\, \tilde{\Theta}(\bar{k}) \qquad k = \sqrt{k^i k^i}$$

$\mu \equiv \frac{\bar{k}\cdot\hat{p}}{k} = \cos\theta$ $\quad\theta$: angle between $\bar{k}$ and $\bar{p}$

$\bar{k}$ aligned with direction in which temperature changes

$\mu = 1$ photon travelling along direction of gradient

$\mu = 0$ photon travelling along direction of constant temperature



Assumption: velocity is irrotational ~~[illegible]~~ $\bar{v} \propto \bar{k} \Rightarrow \tilde{\bar{v}}_b \cdot \hat{p} = \tilde{v}_b \mu$

Define optical depth $\tau(\eta) \equiv \int d\eta'\, a n_e \sigma_T$

late times $\tau \ll 1$

early times $\tau \gg 1$

By definition $\dot{\tau} = \frac{d\tau}{d\eta} = -n_e \sigma_T a$

Fourier transform convention

$$\frac{\partial}{\partial x^i} \longrightarrow i k_i \quad (\equiv i k^i)$$

Fourier transform Boltzmann equation

$$\dot{\Theta} + \underbrace{\hat{p}^i \frac{\partial \Theta}{\partial x^i}}_{ik\mu\tilde{\Theta}} + \dot{\tilde{\Phi}} + \underbrace{\hat{p}^i \frac{\partial \Psi}{\partial x^i}}_{ik_\mu \tilde{\Psi}} = \underbrace{a n_e \sigma_T}_{-\dot{\tau}} \left[\Theta_0 - \Theta + \underbrace{\hat{p} \cdot \vec{v}_b}_{\tilde{v}_b \mu} \right]$$

$$\hookrightarrow \boxed{\dot{\tilde{\Theta}} + ik\mu \tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu \tilde{\Psi} = -\dot{\tau} \left[\hat{\Theta}_0 - \tilde{\Theta} + \mu \tilde{v}_b \right]}$$

Collisional Boltzmann equations for photons
- to 1st order
- in Fourier space
- expressed with temperature perturbation Θ

Boltzmann equations for cold dark matter

Differences with respect to photons:
- DM doesn't interact with other components
  $\hookrightarrow$ NO COLLISION TERM!!!
- DM is non-relativistic

Constraint equation:

$$P^2 = g_{\mu\nu} P^\mu P^\nu = -m^2 \qquad (m \text{ mass of DM})$$

$$E = \sqrt{p^2 + m^2} \quad \text{where} \quad p = g_{ij} P^i P^j \text{ as previously}$$

$\hookrightarrow$ in the massless case $E = p$, which is why E was superfluous for photons once we chose p

So we will use E as one of the variables on which f depends

$f(\bar{x}, t, \hat{p}, E)$

Analogously to photons

$$P^2 = -m^2 = -(1+2\Psi)(P^0)^2 + p^2 \Rightarrow P^0 = \frac{\sqrt{p^2+m^2}}{\sqrt{1+2\Psi}} = \frac{E}{\sqrt{1+2\Psi}} \simeq E(1-\Psi)$$

$$P^i \equiv C\hat{p}^i \Rightarrow p^2 = g_{ij}\hat{p}^i\hat{p}^j C^2 = a^2(1+2\Phi)C^2 \underbrace{\delta_{ij}\hat{p}^i\hat{p}^j}_{1} = a^2(1+2\Phi)C^2$$

$$\Rightarrow p^2 = a^2(1+2\Phi)C^2 \rightarrow C = \frac{p}{a\sqrt{1+2\Phi}} \simeq \frac{p}{a}(1-\Phi)$$

$$\rightarrow P^i = C\hat{p}^i \simeq \frac{p\hat{p}^i(1-\Phi)}{a}$$

$$P^\mu \simeq \left[E(1-\Psi),\ p\hat{p}^i\frac{1-\Phi}{a}\right]$$

$$\frac{df_{dm}}{dt} = \frac{\partial f_{dm}}{\partial t} + \frac{\partial f_{dm}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f_{dm}}{\partial E}\frac{dE}{dt} + \cancel{\frac{\partial f_{dm}}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}}$$

2nd order as for photons

$\frac{dx^i}{dt}$ and $\frac{dE}{dt}$ different compared to photon case

(recall for photons $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[H + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]$)

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{P^i}{P^0} = \frac{p\hat{p}^i(1-\Phi)}{aE(1-\Psi)} = \frac{\hat{p}^i}{a}\frac{p}{E}(1+\Psi-\Phi)$$

But $\frac{\partial f_{dm}}{\partial x^i}$ is already 1st order $\longrightarrow$ $\overset{\text{1st}}{\frac{\partial f_{dm}}{\partial x^i}}\overset{\text{0th}}{\frac{dx^i}{dt}} \simeq \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{dm}}{\partial x^i}$

Similarly $\cancel{\overset{\text{1st}}{\frac{\partial f_{dm}}{\partial \hat{p}^i}}\overset{\text{1st}}{\frac{d\hat{p}^i}{dt}}}$ 2nd order

2nd order so we neglect it

$\frac{dp}{dt}$ identical to the photon case (long calculation through geodesic equation)

$$\frac{dp}{dt} = -p\left[H + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] = -\left[pH + p\frac{\partial\Phi}{\partial t} + \frac{p\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]$$

involved $\frac{dx^i}{dt}$, compensate factor of $\frac{p}{E}$

However here we want to use E, need Jacobian

$$\frac{dE}{dt} = \frac{dE}{dp}\frac{dp}{dt} = \frac{p}{E}\frac{dp}{dt}$$

$$\frac{dE}{dp} = \frac{d}{dp}(p^2+m^2)^{1/2} = \frac{1}{\cancel{2}}\frac{\cancel{2}p}{(p^2+m^2)^{1/2}} = \frac{p}{(p^2+m^2)^{1/2}} = \frac{p}{E}$$

$$\boxed{\frac{df_{dm}}{dt} \simeq \frac{\partial f_{dm}}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f_{dm}}{\partial x^i} - \frac{\partial f_{dm}}{\partial E}\left[\frac{da/dt}{a}\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i p}{a}\frac{\partial\Psi}{\partial x^i}\right] = 0}$$

1st order

Collisionless Boltzmann equation for DM
(reduces to equation for photons as $m \to 0$, $p \to E$)

For photons we proceeded expanding $f \simeq f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta$ BE

For DM we don't know that much about f, and in fact we don't need to!

What we need to know: DM very non-relativistic $v_{dm} \ll 1$

↳ neglect thermal motion of DM

However cannot neglect $\frac{p}{m}$ completely: $\frac{p}{m} \sim v$, velocity flows induced by density perturbations through continuity equation

So: – keep terms linear in $\frac{p}{E}$

– neglect terms quadratic in $\frac{p}{E}$

Don't assume a form for f_{dm}, but take ~~moments~~ moments of the Boltzmann equation

Multiply by $\frac{d^3p}{(2\pi)^3}$, then integrate

$$\cancel{\frac{\partial f_{dm}}{\partial t} + \frac{\hat{p}^i}{a}}$$

Define first two moments

$$n_{dm} \equiv \int \frac{d^3p}{(2\pi)^3} f_{dm}$$

$$v^i \equiv \frac{1}{n_{dm}} \int \frac{d^3p}{(2\pi)^3} \frac{p\hat{p}^i}{E} f_{dm} = \frac{\int \frac{d^3p}{(2\pi)^3} \frac{p\hat{p}^i}{E} f^{dm}}{\int \frac{d^3p}{(2\pi)^3} f_{dm}}$$

$$0 = \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial t} + \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^i}{a} \frac{p}{E} \frac{\partial f_{dm}}{\partial x^i} - \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{E} \frac{\partial f_{dm}}{\partial E} \left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t} \right] - \int \frac{d^3p}{(2\pi)^3} \frac{\hat{p}^i p}{a} \frac{\partial \Psi}{\partial x^i} \frac{\partial f_{dm}}{\partial E}$$

$$\downarrow$$

$$\underbrace{\frac{\partial}{\partial t} \int \frac{d^3p}{(2\pi)^3} f_{dm}}_{\frac{\partial n_{dm}}{\partial t}} + \underbrace{\frac{1}{a} \frac{\partial}{\partial x^i} \int \frac{d^3p}{(2\pi)^3} f_{dm} \frac{\hat{p} p^i}{E}}_{\frac{1}{a}\frac{\partial}{\partial x^i}(n_{dm} v_i)} - \left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t} \right] \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial E} \frac{p^2}{E} - \cancel{\frac{1}{a} \overbrace{\frac{\partial \Psi}{\partial x^i}}^{\text{1st order}} \overbrace{\int \frac{d^3p}{(2\pi)^3} \frac{\hat{p} p^i}{E} \frac{\partial f_{dm}}{\partial E}}^{\text{1st order}}}$$

2nd order

need perturbed part of f_{dm}

$$\int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial E} \frac{p^2}{E} \underset{\frac{dE}{dp} = p}{=} \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial E} \frac{dE}{dp} p = \int \frac{d^3p}{(2\pi)^3} p \frac{\partial f_{dm}}{\partial p} =$$

$$= \frac{4\pi}{(2\pi)^3} \int_0^\infty dp\, p^2 p \frac{\partial f_{dm}}{\partial p} = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp\, p^3 \frac{\partial f_{dm}}{\partial p} \underset{\text{by parts}}{=} -3 \boxed{\frac{4\pi}{(2\pi)^3} \int_0^\infty dp\, p^2 f_{dm}} = -3 n_{dm}$$

$$= \int \frac{d^3p}{(2\pi)^3} f_{dm} = n_{dm}$$

$$\downarrow$$

Cosmological generalization of the continuity equation

$$\boxed{\frac{\partial n_{dm}}{\partial t} + \frac{1}{a} \frac{\partial}{\partial x^i} (n_{dm} v^i) + 3 \left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t} \right] n_{dm} = 0}$$

Now let's collect zeroth and first order terms

$\boxed{\text{0th order}}$

$$\frac{\partial n_{dm}}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}\cancel{(n_{dm} v^i)}^{\text{1st order}} + 3\left[\frac{da/dt}{a} + \cancel{\frac{\partial \Phi}{\partial t}}^{\text{1st order}}\right] n_{dm} = 0$$

$$\hookrightarrow \underbrace{\frac{\partial n_{dm}^{(0)}}{\partial t} + 3\,\frac{da/dt}{a}\, n_{dm}^{(0)}}_{} = 0$$

$$= \frac{1}{a^3}\frac{d}{dt}\left(n_{dm}^{(0)} a^3\right) = \frac{1}{a^3} a^3 \frac{\partial n_{dm}^{(0)}}{\partial t} + \frac{3a^2\dot{a}}{a^3} n_{dm}^{(0)} = \frac{\partial n_{dm}^{(0)}}{\partial t} + 3H n_{dm}^{(0)}$$

$$\frac{d}{dt}\left(n_{dm}^{(0)} a^3\right) = 0 \Longrightarrow \boxed{n_{dm}^{(0)} \propto a^{-3}}$$

confirms that $\rho_{dm} \propto a^{-3}$ as expected for dust component

$\boxed{\text{1st order}}$ (terms $\sim n_{dm} v$, $\sim n_{dm}\Phi \longrightarrow$ set $n_{dm} = n_{dm}^{(0)}$)

$\hookrightarrow$ "small" quantities: Φ, δ, v

$$n_{dm} = n_{dm}^{(0)}\left[1 + \delta(\bar{x}, t)\right] \qquad \text{so} \quad n_{dm}^{(1)} = n_{dm}^{(0)}\delta$$

δ: fractional overdensity $\dfrac{\delta \rho_{dm}}{\rho_{dm}}$

$$\frac{\partial}{\partial t}\left[\cancel{n_{dm}^{(0)}}^{\text{0th}} + n_{dm}^{(0)}\delta\right]_{\text{1st}} + \frac{1}{a}\frac{\partial}{\partial x^i}\left[n_{dm}^{(0)} v^i + \cancel{n_{dm}^{(0)}\delta v_i}^{\text{2nd}}\right]_{\text{1st}} + 3\left[\frac{da/dt}{a} + \frac{\partial\Phi}{\partial t}\right]\left(n_{dm}^{(0)} + n_{dm}^{(0)}\delta\right)$$

$$\simeq \frac{\partial}{\partial t}\left[n_{dm}^{(0)}\delta\right] + \frac{1}{a}\frac{\partial}{\partial x^i}\left[n_{dm}^{(0)} v^i\right] + 3H\left(\cancel{n_{dm}^{(0)}}^{\text{0th}} + n_{dm}^{(0)}\delta\right) + 3\frac{\partial\Phi}{\partial t}\left(n_{dm}^{(0)} + \cancel{n_{dm}^{(0)}\delta}^{\text{2nd}}\right)$$

$$\simeq \frac{\partial n_{dm}^{(0)}}{\partial t}\delta + \frac{\partial\delta}{\partial t} n_{dm}^{(0)} + \frac{1}{a}\frac{\partial n_{dm}^{(0)}}{\partial x^i} v^i + \frac{1}{a} n_{dm}^{(0)}\frac{\partial v^i}{\partial x^i} + 3H n_{dm}^{(0)}\delta + 3\frac{\partial\Phi}{\partial t} n_{dm}^{(0)}$$

$$\frac{\partial}{\partial t}\left[n_{dm}^{(0)}\right] = \frac{\partial}{\partial t}\left[n_{dm,0}\, a^{-3}\right] = n_{dm,0} \times -3\frac{\dot{a}}{a} = -3 n_{dm,0} a^{-3}\frac{\dot{a}}{a} = -3H \underbrace{n_{dm,0}\, a^{-3}}_{n_{dm}^{(0)}} = -3H n_{dm}^{(0)}$$

$$= \cancel{-3H n_{dm}^{(0)}\delta} + n_{dm}^{(0)}\frac{\partial\delta}{\partial t} + \cancel{\frac{v^i}{a}\frac{\partial n_{dm}^{(0)}}{\partial x^i}}^{\text{2nd}} + \frac{1}{a} n_{dm}^{(0)}\frac{\partial v^i}{\partial x^i} + \cancel{3H n_{dm}^{(0)}\delta} + 3\frac{\partial\Phi}{\partial t} n_{dm}^{(0)} =$$

$$= \cancel{n_{dm}^{(0)}} \frac{\partial \delta}{\partial t} + \frac{1}{a} \cancel{n_{dm}^{(0)}} \frac{\partial v^i}{\partial x^i} + 3 \frac{\partial \Phi}{\partial t} \cancel{n_{dm}^{(0)}} = 0$$

$$\Rightarrow \boxed{\frac{\partial \delta}{\partial t} + \frac{1}{a} \frac{\partial v^i}{\partial x^i} + 3 \frac{\partial \Phi}{\partial t} = 0}$$

Boltzmann equation for cold DM overdensity (~~first order~~ / ~~0th moment~~) (~~1st~~ (1st order / 0th moment)

Two variables: δ, v^i

δ ← 0th moment

v^i ← 1st moment

Need another equation

Return to unintegrated Boltzmann equation, but this time take the 1st instead of 0th $\left(\int \times \frac{d^3p}{(2\pi)^3} \right)$ moment

$$\int \times \frac{d^3p}{(2\pi)^3} \frac{p}{E} \hat{p}^j \to \hat{v}^j$$

$$0 = \int \frac{d^3p}{(2\pi)^3} \frac{p\hat{p}_j}{E} \frac{\partial f_{dm}}{\partial t} + \int \frac{d^3p}{(2\pi)^3} \frac{p\hat{p}^j}{E} \frac{\hat{p}^i}{a} \frac{p}{E} \frac{\partial f_{dm}}{\partial x^i} - \int \frac{d^3p}{(2\pi)^3} \frac{p\hat{p}^j}{E} \left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t} \right] \frac{p^2}{E} \frac{\partial f_{dm}}{\partial E}$$

$$- \int \frac{d^3p}{(2\pi)^3} \frac{p\hat{p}^j}{E} \frac{\hat{p}^i}{a} p \frac{\partial \Psi}{\partial x^i} \frac{\partial f_{dm}}{\partial E}$$

$$\underbrace{\frac{\partial}{\partial t} \int \frac{d^3p}{(2\pi)^3} f_{dm} \frac{p\hat{p}^j}{E}}_{\frac{\partial}{\partial t}[n_{dm} \hat{v}_j]} + \cancel{\frac{1}{a} \frac{\partial}{\partial x^i} \int \frac{d^3p}{(2\pi)^3} f_{dm} \frac{p^2 \hat{p}^i \hat{p}^j}{E^2}} - \left[H + \frac{\partial \Phi}{\partial t} \right] \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial E} \frac{p^3 \hat{p}^j}{E^2} - \frac{1}{a} \frac{\partial \Psi}{\partial x^i} \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial E} \frac{p^2 \hat{p}^i \hat{p}^j}{E}$$

$\mathcal{O}\left(\left(\frac{p}{E} \right)^2 \right) \sim v^2$

$$\int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial E} \frac{p^3 \hat{p}^j}{E^2} = \int \frac{d^3p}{(2\pi)^3} \frac{E}{p} \frac{p^3 \hat{p}^j}{E^2} \frac{\partial f_{dm}}{\partial p} = \int \frac{d^3p}{(2\pi)^3} \frac{p^2 \hat{p}^j}{E} \frac{\partial f_{dm}}{\partial p} =$$

$$\left\{ \frac{dE}{dp} = \frac{p}{E} \to \frac{\partial}{\partial E} = \frac{E}{p} \frac{\partial}{\partial p} \right\} = \int \frac{d\Omega \, \hat{p}^j}{(2\pi)^3} \int dp \, \frac{p^4}{E} \frac{\partial f_{dm}}{\partial p} =$$ ← by parts