Summary of Boltzmann equations for γ, DM, b, ν

massless ν basically like γ, just not interacting! → no scattering

+ other effects we glossed over

(Drop ~ from Fourier transforms)

$$\dot\Theta + ik\mu\Theta = -\dot\Phi - ik_\mu\Psi - \dot\tau\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

angular dependence of Compton scattering $|M|^2 \propto |\hat\varepsilon \cdot \hat\varepsilon'|^2$

$$\Pi = \Theta_2 + \Theta_{P2} + \Theta_{P0}$$

temperature field also coupled + polarization field

$$\dot\Theta_P + ik_\mu\Theta_P = -\dot\tau\left[-\Theta_P + \frac{1}{2}\left(1 - P_2(\mu)\right)\Pi\right]$$

evolution of polarization field

$$\dot\delta_{dm} + ikv_{dm} = -3\dot\Phi$$

$$\dot v + \frac{\dot a}{a}v = -ik\Psi$$

$$\dot\delta_b + ikv_b = -3\dot\Phi$$

$$\dot v_b + \frac{\dot a}{a}v_b = -ik\Psi + \frac{\dot\tau}{R}\left[v_b + 3i\Theta_1\right]$$

$$\underbrace{\dot N + ik_\mu N = -\dot\Phi - ik_\mu\Psi}$$

massless neutrinos, without scattering, so equation identical to that for photons, minus scattering

These equation tell us how particle distributions are affected by metric fluctuations. Now we need to know the other way around → equations for Φ, Ψ!
↓
(perturbed) Einstein equations

→ typically expanded in multipoles

$$\Theta_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)$$

e.g. first two moments neglecting scattering

$$\begin{cases} \dot\Theta_0 + k\Theta_1 = -\dot\Phi \\ \dot\Theta_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\Phi \end{cases}$$

THE (PERTURBED) EINSTEIN EQUATIONS

$$\begin{tikzcd}
\text{particle distributions} \arrow[r, "\text{Einstein equations}"] & \text{perturbations in the gravitational field} \\
f \Rightarrow \boxed{\Theta, \delta, v, N} & \boxed{\Phi, \Psi} \arrow[l, "\text{Boltzmann equations}"]
\end{tikzcd}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

→ expand perturbatively around zero-order homogeneous solution

LHS 3 steps:

1) $\Gamma^{\mu}_{\alpha\beta}$ for perturbed metric $ds^2 = -(1+2\Psi)dt^2 + a^2(1+2\Phi)\delta_{ij}dx^i dx^j$

2) $R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$

3) $R = g^{\mu\nu}R_{\mu\nu}$

Perturbed Ricci tensor and scalar (for scalar perturbations)

Christoffel symbols

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]$$

0th order

$\Gamma^{0}_{00} = 0$ $\Gamma^{0}_{0i} = \Gamma^{0}_{i0} = 0$ $\Gamma^{0}_{ij} = \delta_{ij}\dot{a}a$ $\Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \delta_{ij}\frac{\dot{a}}{a}$ all other Γ's $= 0$

1st order: only take terms linear in Φ or Ψ

Start from $\Gamma^{0}_{\mu\nu}$

$$\Gamma^{0}_{\mu\nu} = \frac{g^{0\alpha}}{2}\left[\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right]$$

$$\Gamma^{0}_{\mu\nu} = \frac{g^{0\boxed{\alpha}}}{2}\left[\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right]$$

$\rightarrow \alpha = 0$

$$g^{00} = (g_{00})^{-1} = \cancel{(-1-2\psi)}\left[-(1+2\psi)\right]^{-1} \simeq \boxed{-1+2\psi}$$

$$\boxed{\Gamma^{0}_{\mu\nu}} \simeq \frac{-1+2\psi}{2}\left[\frac{\partial g_{0\mu}}{\partial x^{\nu}} + \frac{\partial g_{0\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial t}\right]$$

Look at components

- $\mu=0,\ \nu=0$

$$\Gamma^{0}_{00} \simeq \frac{-1+2\psi}{2}\left[\frac{\partial g_{00}}{\partial t} + \cancel{\frac{\partial g_{00}}{\partial t}} - \cancel{\frac{\partial g_{00}}{\partial t}}\right] \simeq \left(\frac{-1+2\psi}{2}\right)\frac{\partial g_{00}}{\partial t}$$

$$g_{00} = -1-2\psi \rightarrow \frac{\partial g_{00}}{\partial t} = -2\frac{\partial\psi}{\partial t} \longrightarrow \Gamma^{0}_{00} \simeq \left(\frac{-1+2\psi}{2}\right) \times -2\frac{\partial\psi}{\partial t}$$

$$\rightarrow \boxed{\Gamma^{0}_{00} \simeq \frac{\partial\psi}{\partial t}}$$

drop 2nd order term $\propto \psi\frac{\partial\psi}{\partial t}$

- $\mu=0,\ \nu=i$ or $\mu=i,\ \nu=0$ (Γ symmetric)

$$\cancel{\Gamma^{0}_{0i} = \frac{g^{0\alpha}}{2}\left[\partial g\right.}\qquad \Gamma^{0}_{0i} \simeq \frac{-1+2\psi}{2}\left[\frac{\partial g_{00}}{\partial x^{i}} + \cancelto{0}{\frac{\partial g_{0i}}{\partial t}} - \cancelto{0}{\frac{\partial g_{0i}}{\partial t}}\right]$$

$$= \left(\frac{-1+2\psi}{2}\right)\frac{\partial g_{00}}{\partial x^{i}} = \left(\frac{-1+2\psi}{2}\right) \times -2\frac{\partial\psi}{\partial x^{i}} \simeq \frac{\partial\psi}{\partial x^{i}} \longrightarrow ik_{i}\psi$$

$\frac{\partial g_{00}}{\partial x^{i}} = -2\frac{\partial\psi}{\partial x^{i}}$

drop 2nd order term $\propto \psi\frac{\partial\psi}{\partial x^{i}}$

Fourier Space

- $\mu=i,\ \nu=j$

$$\Gamma^{0}_{ij} \simeq \frac{-1+2\psi}{2}\left[\cancelto{0}{\frac{\partial g_{0i}}{\partial x^{j}}} + \cancelto{0}{\frac{\partial g_{0j}}{\partial x^{i}}} - \frac{\partial g_{ij}}{\partial t}\right] = \frac{1-2\psi}{2}\frac{\partial}{\partial t}\left[\delta_{ij}a^{2}(1+2\Phi)\right]$$

$$\frac{\partial}{\partial t}\left[\delta_{ij}a^{2}(1+2\Phi)\right] = \delta_{ij}2a\frac{da}{dt}(1+2\Phi) + 2\delta_{ij}a^{2}\frac{\partial\Phi}{\partial t} = \delta_{ij}a^{2}\left(2\frac{1}{a}\frac{da}{dt}(1+2\Phi) + 2\frac{\partial\Phi}{\partial t}\right) =$$

$$= \cancel{\delta_{ij} a^2 \left(2H(1+2\Phi) + 2\frac{\partial\Phi}{\partial t}\right)} \qquad = \delta_{ij} a^2 \left(2H(1+2\Phi) + 2\frac{\partial\Phi}{\partial t}\right)$$

$$\Gamma^0_{ij} \simeq \frac{1-2\psi}{2}\frac{\partial}{\partial t}\left[\delta_{ij}a^2(1+2\Phi)\right] = \frac{(1-2\psi)}{2}\delta_{ij}a^2\left(2H(1+2\Phi) + 2\frac{\partial\Phi}{\partial t}\right) =$$

$$= \delta_{ij}\frac{a^2}{2}\left[2H + 4H\Phi + 2\frac{\partial\Phi}{\partial t} - 4H\psi - \cancel{4H\psi\Phi} - \cancel{4\psi\frac{\partial\Phi}{\partial t}}\right] =$$

2nd order 2nd order

$$= \delta_{ij}\frac{a^2}{2}\left[2H + 4H(\Phi-\psi) + 2\frac{\partial\Phi}{\partial t}\right] = \delta_{ij}a^2\left[H + 2H(\Phi-\psi) + \frac{\partial\Phi}{\partial t}\right]$$

Other Christoffel symbols, exercise

In summary

$$\Gamma^0_{00} = \frac{\partial\psi}{\partial t} \qquad \Gamma^0_{0i} = \Gamma^0_{i0} = ik_i\psi \qquad \Gamma^0_{ij} = \delta_{ij}a^2\left[H + 2H(\Phi-\psi) + \frac{\partial\Phi}{\partial t}\right]$$

$$\Gamma^i_{00} = \frac{ik^i}{a^2}\psi \qquad \Gamma^i_{j0} = \Gamma^i_{0j} = \delta_{ij}\left(H + \frac{\partial\Phi}{\partial t}\right)$$

only non-vanishing zero-order components

$$\Gamma^i_{jk} = i\Phi\left(\delta_{ij}k_k + \delta_{ik}k_j - \delta_{jk}k_i\right)$$

note: $\delta_{ij} = \delta^{ij}$ $k_i = k^i$

Ricci tensor

$$R_{00} = \partial_\alpha\Gamma^\alpha_{00} - \partial_0\Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha}$$

consider $\alpha = 0$ $\cancel{\beta}$ $\cancel{\partial_0\Gamma^0_{00}} - \cancel{\partial_0\Gamma^0_0} + \cancel{\Gamma^0_{\beta\alpha}\Gamma^\beta_{00}} - \Gamma^0_{\cancel{\beta 0}}\cancel{\Gamma^\beta_{00}}$

$\hookrightarrow$ so ~~Φ~~ sum over index α only contributes when $\alpha = i$

$$R_{00} = \underset{\textcircled{1}}{\partial_i\Gamma^i_{00}} - \underset{\textcircled{2}}{\partial_0\Gamma^i_{0i}} + \underset{\textcircled{3}}{\Gamma^i_{\beta i}\Gamma^\beta_{00}} - \underset{\textcircled{4}}{\Gamma^i_{\beta 0}\Gamma^\beta_{0i}}$$

$$\textcircled{1}\quad \partial_i\Gamma^i_{00} \longrightarrow ik_i\Gamma^i_{00} = ik_i\frac{ik^i}{a^2}\psi = -\frac{k^2}{a^2}\psi$$

Fourier space

$$\partial_i\Gamma^i_{00} = -\frac{k^2}{a^2}\psi$$

② $$-\partial_0 \Gamma^i_{0i} = -\frac{\partial}{\partial t}\left[\underbrace{\delta_{ii}}_{\text{"}\times 3\text{"}}\left(H+\frac{\partial\Phi}{\partial t}\right)\right] = -3\frac{\partial}{\partial t}\left(H+\frac{\partial\Phi}{\partial t}\right) *$$

$$\frac{dH}{dt} = \frac{d}{dt}\left(\frac{1}{a}\frac{da}{dt}\right) = \cancel{\frac{1}{a}\frac{d^2a}{dt^2}} = \frac{a\, d^2a/dt^2 - (da/dt)^2}{a^2} = \frac{1}{a}\frac{d^2a}{dt^2} - \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{1}{a}\frac{d^2a}{dt^2} - H^2$$

$$* = -3\left(\frac{d^2a/dt^2}{a} - H^2 + \frac{\partial^2\Phi}{\partial t^2}\right)$$

③ $$\underbrace{\Gamma^i_{i\beta}}_{\text{only keep 0th order part}}\underbrace{\Gamma^\beta_{00}}_{\text{1st order}} \qquad \Gamma^\beta_{00} = \frac{ik^\beta}{a^2}\Psi \quad (\beta=i), \quad \frac{\partial\Psi}{\partial t} \quad (\beta=0) \longrightarrow \text{1st order!}$$

only keep 0th order part →

$$\left.\begin{aligned} &\Gamma^i_{i\beta} \sim i\Phi[\delta K + \delta K - \delta K] \quad \text{always 1st order} \\ &\Gamma^i_{i0} = \delta_{ii}\left(H + \frac{\partial\Phi}{\partial t}\right) \quad \text{has 0th order part} \end{aligned}\right\} \text{need } \beta = 0!$$

$$\Rightarrow \Gamma^i_{i\beta}\Gamma^\beta_{00} \simeq \cancel{\delta_{ii} H}\, \Gamma^i_{i0}\Gamma^0_{00} \simeq \delta_{ii} H \frac{\partial\Psi}{\partial t} = 3H\frac{\partial\Psi}{\partial t}$$

$\Gamma^i_{i\beta}$ 0th-order part, $\beta=0$

δ_{ii} = "×3"

④ $$-\Gamma^i_{\beta 0}\Gamma^\beta_{0i} \qquad \beta=0 \sim \underbrace{\Gamma^i_{00}}_{\substack{\text{1st order} \\ \frac{ik^i\Psi}{a^2}}}\underbrace{\Gamma^0_{0i}}_{\substack{\text{1st order} \\ ik^i\Psi}} \sim \text{2nd order}$$

↓ must be $\beta = j$!

$\beta = j$ →

$$\simeq -\Gamma^i_{j0}\Gamma^j_{0i} = -\delta_{ij}\left(H+\frac{\partial\Phi}{\partial t}\right)\delta_{ji}\left(H+\frac{\partial\Phi}{\partial t}\right) =$$

$\delta_{ij}\delta_{ji} = \delta_{ij}$

$$= -\delta_{ij}\left(H+\frac{\partial\Phi}{\partial t}\right)\left(H+\frac{\partial\Phi}{\partial t}\right) \simeq -3\left(H^2 + 2H\frac{\partial\Phi}{\partial t}\right)$$

δ_{ij} = "×3"

Putting everything together

$$R_{00} = ① + ② + ③ + ④ = -\frac{k^2}{a^2}\Psi - 3\left(\frac{d^2a/dt^2}{a} - H^2 + \frac{\partial^2\Phi}{\partial t^2}\right) + 3H\frac{\partial\Psi}{\partial t} - 3\left(H^2 + 2H\frac{\partial\Phi}{\partial t}\right) =$$

$$= -\frac{k^2}{a^2}\Psi - 3\frac{d^2a/dt^2}{a} \cancel{+3H^2} - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\frac{\partial\Psi}{\partial t} \cancel{-3H^2} - 6H\frac{\partial\Phi}{\partial t} =$$

$$\boxed{\begin{aligned}\Rightarrow\ R_{00} &= -3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\Psi - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\left(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right)\\ &\text{(exercise)}\\ R_{ij} &= \delta_{ij}\left[\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right)(1+2\Phi-2\Psi) + a^2H\left(6\frac{\partial\Phi}{\partial t} - \frac{\partial\Psi}{\partial t}\right) + a^2\frac{\partial^2\Phi}{\partial t^2} + k^2\Phi\right]\\ &\quad + k_ik_j(\Phi+\Psi)\end{aligned}}$$

$$\overset{\text{Ricci scalar}}{R} = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij} =$$

$$g^{00} \simeq [-(1+2\Psi)]^{-1} \simeq -1+2\Psi$$

$$g^{ij} = \left[\delta_{ij}a^2(1+2\Phi)\right]^{-1} \simeq \frac{\delta_{ij}}{a^2}(1-2\Phi)$$

$$= \underbrace{R_0}_{\text{0th order}} + \underbrace{\delta R}_{\text{1st order}}$$

$$R = g^{00}R_{00} + g^{ij}R_{ij} = (-1+2\Psi)\left[-3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\Psi - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\left(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right)\right]$$

$$+ \underset{\delta_{ij}}{\cancel{\delta}}\left(\frac{1-2\Phi}{a^2}\right)\Big[\cancel{\delta}\ 3\Big\{\left[\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right)(1-2\Phi-2\Psi) + a^2H\left(6\frac{\partial\Phi}{\partial t} - \frac{\partial\Psi}{\partial t}\right) + a^2\frac{\partial^2\Phi}{\partial t^2} + k^2\Phi\right\}$$

$$\delta_{ij} \qquad + k^2(\Phi+\Psi)\Big]$$

Look at 0th order pieces

$$R_0 = -1 \times -3\frac{1}{a}\frac{d^2a}{dt^2} + \frac{1}{a^2}\times 3\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right) = 3\frac{1}{a}\frac{d^2a}{dt^2} + 6H^2 + 3\frac{1}{a}\frac{d^2a}{dt^2} =$$

$$= 6\left(H^2 + \frac{d^2a/dt^2}{a}\right) \checkmark \longrightarrow$$ already seen in 2nd Friedmann equation!

$$\delta R = \left(g^{00}R_{00}\right)_{1st} + \left(g^{ij}R_{ij}\right)_{1st}$$

$$\left(g^{00}R_{00}\right)_{1st} = \overset{\text{0th}}{\cancel{3\frac{d^2a}{dt^2}}} + \frac{k^2}{a^2}\Psi + \frac{3\partial^2\Phi}{\partial t^2} - 3H\frac{\partial\Psi}{\partial t} + 6H\frac{\partial\Phi}{\partial t} - 6\Psi\frac{d^2a/dt^2}{a} - \overset{\text{2nd}}{\cancel{2\frac{k^2}{a^2}\Psi^2}} - \overset{\text{2nd}}{\cancel{6\Psi\frac{\partial^2\Phi}{\partial t^2}}}$$

$$+ \overset{\text{2nd}}{\cancel{6H\Psi\frac{\partial\Psi}{\partial t}}} - \overset{\text{2nd}}{\cancel{12H\Psi\frac{\partial\Phi}{\partial t}}} \simeq -6\Psi\frac{d^2a/dt^2}{a} + \frac{k^2}{a^2}\Psi + 3\frac{\partial^2\Phi}{\partial t^2} - 3H\left(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right)$$

$$\left(g^{ij}R_{ij}\right)_{1st} = \left(\frac{1-2\Phi}{a^2}\right)\Big[6a^2H^2 + 12a^2H^2\Psi - 12a^2H^2\Psi + 3a\frac{d^2a}{dt^2} + 6a\frac{d^2a}{dt^2}\Phi - 6a\frac{d^2a}{dt^2}\Psi$$

$$+ 18a^2H\frac{\partial\Phi}{\partial t} - 3a^2H\frac{\partial\Psi}{\partial t} + 3a^2\frac{\partial^2\Phi}{\partial t^2} + 3k^2\Phi + k^2\Phi + k^2\Psi\Big] =$$

$$= \left(\frac{1-2\Phi}{a^2}\right)\Big[-6\Psi\left(a\frac{d^2a}{dt^2} + 2a^2H^2\right) + 3H\left(6a^2\frac{\partial\Phi}{\partial t} - a^2\frac{\partial\Psi}{\partial t}\right)$$

$$+ 3a\frac{d^2a}{dt^2}(1+2\Phi) + 6a^2H^2(1+2\Phi)$$

$$+ 3a^2\frac{\partial^2\Phi}{\partial t^2} + 4k^2\Phi + k^2\Psi\Big] \simeq$$

$(1+2\Phi)(1-2\Phi) \simeq 1$, left with ~~$3a\frac{d^2a}{dt^2} + 6a^2H^2$, both 0th order~~

all 1st order, multiply by $\frac{1}{a^2}$

$$\simeq -6\Psi\left(2H^2 + \frac{d^2a/dt^2}{a}\right) + 3H\left(6\frac{\partial\Phi}{\partial t} - \frac{\partial\Psi}{\partial t}\right) + 3\frac{\partial^2\Phi}{\partial t^2} + 4\frac{k^2}{a^2}\Phi + \frac{k^2}{a^2}\Psi$$

Putting everything together...

$$\delta R = -6\Psi\frac{d^2a/dt^2}{a} + \frac{k^2}{a^2}\Psi + 3\frac{\partial^2\Phi}{\partial t^2} - 3H\left(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right) - 6\Psi\left(2H^2 + \frac{d^2a/dt^2}{a}\right)$$

$$+ 3H\left(6\frac{\partial\Phi}{\partial t} - \frac{\partial\Psi}{\partial t}\right) + 3\frac{\partial^2\Phi}{\partial t^2} + 4\frac{k^2}{a^2}\Phi + \frac{k^2}{a^2}\Psi =$$

$$= -12\Psi\left(H^2 + \frac{d^2a/dt^2}{a}\right) + \frac{2k^2}{a^2}\Psi + 6\frac{\partial^2\Phi}{\partial t^2} - 6H\left(\frac{\partial\Psi}{\partial t} - 4\frac{\partial\Phi}{\partial t}\right) + 4\frac{k^2}{a^2}\Phi$$

$$\boxed{\delta R = -12\Psi\left(H^2 + \frac{d^2a/dt^2}{a}\right) + \frac{2k^2}{a^2}\Psi + 6\frac{\partial^2\Phi}{\partial t^2} - 6H\left(\frac{\partial\Psi}{\partial t} - 4\frac{\partial\Phi}{\partial t}\right) + 4\frac{k^2}{a^2}\Phi}$$

Two components of the Einstein equations

Now derive evolution equations for Φ, Ψ

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

↳ 10 components, but we need only 2! (for scalar)

So let's look only at 2 components:

- time-time (00)
- longitudinal, traceless part of space-space $(\sim P_{ij}{}^{ij})$

$$\boxed{G^{0}{}_{0}} = g^{00}\left[R_{00} - \frac{1}{2}g_{00}R\right] = g^{00}R_{00} - \frac{1}{2}\underbrace{g^{00}g_{00}}_{=1}R \simeq (-1+2\Psi)R_{00} - \frac{R}{2}$$

↳ in hindsight simplifies $T^{\mu}{}_{\nu}$

Already computed R_{00} ~~earlier~~ and δR earlier

$$R_{00} = -3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\Psi - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\left(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right)$$

$$(\delta R) = -12\Psi\left(H^2 + \frac{d^2a/dt^2}{a}\right) + \frac{2k^2}{a^2}\Psi + 6\frac{\partial^2\Phi}{\partial t^2} - 6H\left(\frac{\partial\Psi}{\partial t} - 4\frac{\partial\Phi}{\partial t}\right) + \frac{4k^2\Phi}{a^2}$$

1st order

$$\Rightarrow (\delta G^{0}{}_{0})_{1st} = \text{[illegible]} \left[(-1+2\Psi)R_{00}\right]_{1st} - \frac{(\delta R)_{1st}}{2}$$

$$\simeq \underbrace{2\Psi \times \cancel{-3\frac{d^2a/dt^2}{a}}}_{\times 2\Psi} \cancel{+\frac{k^2}{a^2}\Psi} \underbrace{\cancel{+3\frac{\partial^2\Phi}{\partial t^2}} - 3H\left(\cancel{\frac{\partial\Psi}{\partial t}} - \frac{2\partial\Phi}{\partial t}\right)}_{\times -1 \text{ since } \Psi \text{ already is 1st order}} + 6\Psi\left(H^2 + \cancel{\frac{d^2a/dt^2}{a}}\right) \cancel{-\frac{k^2}{a^2}\Psi}$$

$$\cancel{-3\frac{\partial^2\Phi}{\partial t^2}} + 3H\left(\cancel{\frac{\partial\Psi}{\partial t}} - 4\frac{\partial\Phi}{\partial t}\right) - 2k^2\frac{\Phi}{a^2} =$$

$$= -6H\frac{\partial\Phi}{\partial t} + 6\Psi H^2 - 2k^2\frac{\Phi}{a^2}$$

$$\rightarrow \boxed{(\delta G^{0}{}_{0})_{1st} \simeq -6H\frac{\partial\Phi}{\partial t} + 6\Psi H^2 - 2k^2\frac{\Phi}{a^2}} \left[= 8\pi G (T^{0}{}_{0})_{1st}\right]$$

Now need 1st-order part of $T^0{}_0$

Recall: $-T^0{}_0$ energy density of all particles ($\sim \int f$)

$\hookrightarrow T^\mu{}_\nu = \mathrm{diag}(-\rho, \mathcal{P}, \mathcal{P}, \mathcal{P})$

So

$$T^0{}_0(\bar{x},t) = -\underbrace{\sum_i}_{\text{all species}} g_i \int \frac{d^3p}{(2\pi)^3} E_i(p)\, \underbrace{f_i(\bar{p},\bar{x},t)}$$

look at 1st-order part for γ, ν, DM, b

DM, baryons:

By <u>definition</u> $T^0{}_0(\bar{x},t) = -\sum_i \bar{\rho}_i(1+\delta_i)$ $\quad i = \text{DM}, b$

$\hookrightarrow \delta T^0{}_0 = -\rho_i \delta_i$

Photons:

Recall definition of Θ

$$f = \left[\exp\left\{\frac{p}{T(t)[1+\Theta(\bar{x},\hat{p},t)]}\right\} - 1\right]^{-1} \underset{\Theta \ll 1}{\simeq} f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta$$

$$\rightarrow \left[T^0{}_0(\bar{x},t)\right]_\gamma = \underbrace{-2}_{g_\gamma} \int \frac{d^3p}{(2\pi)^3}\, p \left[\underset{①}{f^{(0)}} - \underset{②}{p\frac{\partial f^{(0)}}{\partial p}\Theta}\right]$$

① $\displaystyle -2\int \frac{d^3p}{(2\pi)^3} p f^{(0)} = -\rho_\gamma$

② $\displaystyle 2\int \frac{d^3p}{(2\pi)^3} p^2 \frac{\partial f^{(0)}}{\partial p}\Theta =$ $\qquad$ Recall $\displaystyle \Theta_0 = \frac{1}{4\pi}\int d\Omega' \,\Theta(\hat{p}', \bar{x}, t)$

$$= 2\underbrace{\int d\Omega\, \Theta}_{4\pi\Theta_0} \int \frac{dp}{(2\pi)^3} p^4 \frac{\partial f^{(0)}}{\partial p} \underset{\uparrow \text{by parts}}{=} \cancel{4\pi\Theta_0}\, \Theta_0\, 4\pi\left(-2\int \frac{dp}{(2\pi)^3} 4p^3 f^{(0)}\right) =$$

$$= -4\Theta_0\left[2\int \frac{dp\, p^3}{(2\pi)^3} 4\pi f^{(0)}\right] = -4\rho_\gamma\Theta_0 .$$

$$\hookrightarrow 2\int \frac{d^3p}{(2\pi)^3} p f^{(0)} = \rho_\gamma$$