$$= -\cancel{\int \frac{d\Omega \, \hat{p}^j}{(2\pi)^3} \, f_{dm} \frac{\partial}{\partial p}\left(\frac{p^4}{E}\right)}$$

$$\frac{\partial}{\partial p}\left(\frac{p^4}{E}\right) = \frac{4p^3 E - p^4 \frac{dE}{dp}}{E^2} = \frac{4p^3 E - p^4 \frac{p}{E}}{E^2} = \frac{4p^3 E - \frac{p^5}{E}}{E^2} = 4\frac{p^3}{E} - \frac{p^5}{E^3}$$

$$= -\int \frac{d\Omega \, \hat{p}^j}{(2\pi)^3} \int dp \, f_{dm} \frac{\partial}{\partial p}\left(\frac{p^4}{E}\right) = -\int \frac{d\Omega \, \hat{p}^j}{(2\pi)^3} \int_0^\infty dp \, f_{dm} \left(\frac{4p^3}{E} - \cancel{\frac{p^5}{E^3}}\right) \quad O\left(\left(\frac{p}{E}\right)^3\right) \sim v^3 !!!$$

$$\simeq -4 \int \frac{d\Omega \, \hat{p}^j}{(2\pi)^3} \int_0^\infty dp \, \frac{p^3}{E} f_{dm} = -4 \underbrace{\int \frac{d\Omega}{(2\pi)^3} \int_0^\infty dp \, p^2}_{\int \frac{d^3p}{(2\pi)^3}} \frac{p}{E} \hat{p}^j f_{dm} = \underbrace{-4 \int \frac{d^3p}{(2\pi)^3} \frac{p \hat{p}^j}{E} f_{dm}}_{n_{dm} v^j} =$$

$$= -4 n_{dm} v_j$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial E} \frac{\hat{p}^i \hat{p}^j p^2}{E} = \int \frac{d^3p}{(2\pi)^3} \frac{E}{p} \frac{\partial f_{dm}}{\partial p} \frac{\hat{p}^i \hat{p}^j p^2}{E} = \int \frac{d^3p}{(2\pi)^3} \hat{p}^i \hat{p}^j p \frac{\partial f_{dm}}{\partial p} =$$

$$= \underbrace{\int d\Omega \, \hat{p}^i \hat{p}^j}_{\delta^{ij} \frac{4\pi}{3}} \int \frac{dp}{(2\pi)^3} p^3 \frac{\partial f_{dm}}{\partial p} = -\delta^{ij} \frac{4\pi}{3} \int \frac{dp}{8\pi^3} 3p^2 f_{dm} = -\delta^{ij} \frac{\cancel{4\pi}}{\cancel{3} \times 8\pi^3} \cancel{3} \int dp \, p^2 f_{dm} =$$

$$= -\frac{\delta^{ij}}{2\pi^2} \int dp \, p^2 f_{dm} \longrightarrow \underbrace{\int \frac{d^3p}{(2\pi)^3} f_{dm}}_{n_{dm}} = \frac{4\pi}{8\pi^3} \int dp \, p^2 f_{dm} = \frac{1}{2\pi^2} \int dp \, p^2 f_{dm}$$

$$= -\delta^{ij} n_{dm}$$

$$\longrightarrow -\frac{1}{a} \frac{\partial \Psi}{\partial x^i} \int \frac{d^3p}{(2\pi)^3} \frac{\partial f_{dm}}{\partial E} \hat{p} \frac{p^2 \hat{p}^i \hat{p}^j}{E} = -\frac{1}{a} \frac{\partial \Psi}{\partial x^i} \times -\delta^{ij} n_{dm} = \frac{n_{dm}}{a} \frac{\partial \Psi}{\partial x^j}$$

Putting everything together, 1st moment of Boltzmann equation:

2nd order

$$\frac{\partial}{\partial t}(n_{dm} v^j) - 4 n_{dm} v_j \times \left[\frac{da/dt}{a} + \cancel{\frac{\partial \Phi}{\partial t}}\right] + \frac{n_{dm}}{a} \frac{\partial \Psi}{\partial x^j} \simeq \frac{\partial}{\partial t}(n_{dm} v_j) + 4H n_{dm} v_j + \frac{n_{dm}}{a} \frac{\partial \Psi}{\partial x^j} = 0$$

v, Ψ 1st order, so there are no 0th order terms

$\hookrightarrow$ Can set $n_{dm} = n_{dm}^{(0)}$ everywhere (otherwise we get 2nd order terms)

$$\frac{\partial}{\partial t}(n_{dm} v^J) + 4H n_{dm} v^J + \frac{n_{dm}}{a}\frac{\partial \Psi}{\partial x^J} = 0$$

$\downarrow$

$$\frac{\partial}{\partial t}(n_{dm}^{(0)} v^J) + 4H n_{dm}^{(0)} v^J + \frac{n_{dm}^{(0)}}{a}\frac{\partial \Psi}{\partial x^J} = 0$$

$\downarrow$ $n_{dm}^{(0)} \propto a^{-3} \rightarrow \frac{\partial}{\partial t} n_{dm}^{(0)} = -3H n_{dm}^{(0)}$

$$\cancel{-3H} \cancel{n_{dm}^{(0)}} v^J + \cancel{n_{dm}^{(0)}} \frac{\partial v^J}{\partial t} + \cancel{4}H \cancel{n_{dm}^{(0)}} v^J + \frac{\cancel{n_{dm}^{(0)}}}{a}\frac{\partial \Psi}{\partial x^J} = 0$$

$\Downarrow$

$$\boxed{\frac{\partial v^J}{\partial t} + H v^J + \frac{1}{a}\frac{\partial \Psi}{\partial x^J} = 0}$$

* Boltzmann equation for cold DM velocity (1st order / 1st moment)

$\hookrightarrow$ no $(\vec{v}\cdot\vec{\nabla})\vec{v}$ term since it is 2nd order!

Note: the equation for the nth moment depends on the (N+1)th moment $\rightarrow$ infinite hierarchy (in principle)

$\delta \rightarrow v$

$v \rightarrow$ quadrupole / anisotropic stress? $\rightarrow$ Dropped terms $\mathcal{O}\left(\left(\frac{p}{E}\right)^2\right)$!!! Because DM is cold

sufficient to ~~characterize~~ CDM !!

$\rightarrow$ Generic feature of integrating Boltzmann equation to get fluid equations

~~* Conformal~~ time

Equations for δ and v

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0$$

conformal time

$$d\eta = \frac{dt}{a} \rightarrow \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta} = \frac{1}{a}\,\dot{}$$

$$\Longrightarrow \quad \dot{\delta} + \frac{\partial v^i}{\partial x^i} + 3\dot{\Phi} = 0$$

$$\frac{\partial v^J}{\partial t} + H v^J + \frac{1}{a}\frac{\partial \Psi}{\partial x^J} = 0$$

$$\dot{v}_J + \underbrace{\frac{\dot{a}}{a} v_J}_{\mathcal{H} v_J} + \frac{\partial \Psi}{\partial x^J} = 0$$

As usual take Fourier transform, assume irrotational velocity $\left(\bar{v} \propto \hat{k} \rightarrow \tilde{v}^i = \left(\frac{k^i}{k}\right)\tilde{v} \rightarrow \left(\frac{\partial \tilde{v}}{\partial x^i}\right) = ik\tilde{v}\right)$

$$\rightarrow \boxed{\begin{aligned} &\dot{\tilde{\delta}} + ik\tilde{v} + 3\dot{\tilde{\Phi}} = 0 \\ &\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0 \end{aligned}}$$

Collisionless Boltzmann equation for DM:
- to 1st order
- in Fourier space
- expressed in terms of 2 lowest order moments δ, v (since DM is cold!!)

Boltzmann equations for baryons

Baryons: $\underbrace{\text{electrons} + \text{protons}}_{}$ (bad nomenclature)

coupled by Coulomb scattering $e + p \rightarrow e + p$
rate always larger than H

$$\rightarrow \frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}} \equiv \delta_b$$

strong scattering rate forces common overdensity

$$\bar{v}_e = \bar{v}_p \equiv \bar{v}_b$$

similarly for velocity

So similarly to ~~dm~~ dark matter we want the equations for δ_b, v_b

Start from unintegrated equations for e, p:

$$\frac{df_e}{dt}(\bar{x}, \bar{q}, t) = \underbrace{\langle C_{ep} \rangle_{QQ'q'}}_{} + \langle C_{e\gamma} \rangle_{pp'q'} \rightarrow \text{Compton scattering}$$

$e^-(\bar{q}) + \gamma(\bar{p}) \rightarrow e^-(\bar{q}') + \gamma(\bar{p}')$

$$\frac{df_p}{dt}(\bar{x}, \bar{Q}, t) = \langle C_{ep} \rangle_{qq'Q'}$$

Coulomb scattering
$e^-(\bar{q}) + p^+(\bar{Q}) \rightarrow e^-(\bar{q}') + p^+(\bar{Q}')$

Compact notation (compare Boltzmann equations for photons)

Unintegrated part of collision term

$$C_{e\gamma} \equiv (2\pi)^4 \delta^4(p+q-p'-q') \frac{|M|^2_{Compton}}{8 E_\gamma(p) E_\gamma(p') E_e(q') E_e(q)} \left[f_e(q') f_\gamma(p') - f_e(q) f_\gamma(p) \right]$$

$\to 2^3$

$$\langle (\ldots) \rangle_{pp'q'} \equiv \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} (\ldots)$$

Similarly for C_{ep}: $|M|^2_{Compton} \to |M|^2_{Coulomb}$ and integrated momenta are different

$\langle C_{p\gamma} \rangle$ neglected since $\sigma \propto \frac{1}{m_p^2}$, so interactions of combined electron-proton fluid with photons driven by Compton scattering between electrons and photons

(Also neglecting ionization and recombination terms)

Now multiply $\frac{df_e}{dt}$ equation by $\frac{d^3q}{(2\pi)^3}$ and integrate $\to$ LHS identical to DM, recall:

$$\frac{df_e}{dt} = \frac{\partial f_e}{\partial t} + \frac{\partial f_e}{\partial x^i} \frac{\hat{p}^i}{a} \frac{p}{E} - \frac{\partial f_e}{\partial E} \left[H \frac{p^2}{E} + \frac{p^2}{E} \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i p}{a} \frac{\partial \Psi}{\partial x^i} \right] = 0$$

$$\Big\downarrow \int \times \frac{d^3q}{(2\pi)^3}$$

$$\cancel{\frac{\partial n_e}{\partial t} + \frac{1}{a} \frac{\partial (n_e v_b^i)}{\partial x^i} + 3 \left[H + \frac{\partial \Phi}{\partial t} \right] n_e = \langle C_{ep} \rangle_{QQ'q'} + \langle C_{e\gamma} \rangle_{pp'q'}}$$

$$\frac{\partial n_e}{\partial t} + \frac{1}{a} \frac{\partial (n_e v_b^i)}{\partial x^i} + 3 \left[H + \frac{\partial \Phi}{\partial t} \right] n_e = \cancelto{0}{\langle C_{ep} \rangle_{QQ'q'}} + \cancelto{0}{\langle C_{e\gamma} \rangle_{pp'q'}}$$

$\equiv 0$

mathematical physical identity, not approximation why?, Take e.g.

$$\langle C_{ep} \rangle_{QQ'q'q}$$

$\langle \dots \rangle_{QQ'q'q}$ symmetric in $Q \leftrightarrow Q'$ $q \leftrightarrow q'$

C_{ep} antisymmetric in $Q \leftrightarrow Q'$ $q \leftrightarrow q'$ because of

$$f_e(q) f_p(Q) - f_e(q') f_p(Q')$$

So integral is identically 0!

Physically speaking, Coulomb scattering conserves electron number, so ~~cannot~~ contribute to $\frac{\partial n_e}{\partial t}$

$\int f_e(q') f_p(Q')$ $\propto$ # electrons produced in Coulomb scattering

$$=$$

$\int f_e(q) f_p(Q)$ $\propto$ # electrons lost in Coulomb scattering

Similarly for Compton scattering

True in general: multiplying unintegrated collision term by conserved quantity and integrating $\longrightarrow 0$

So for δ_b we have the exact same equation as cold DM.

Using conformal time and going to Fourier space:

$$\boxed{\dot{\tilde{\delta}}_b + iK \tilde{v}_b + 3\dot{\tilde{\Phi}} = 0}$$

Baltzmann equation for baryon overdensity (1st order/0th moment)

As with DM we also need an equation for the velocity, take first moments of $\frac{df_e}{dt}$ and $\frac{df_p}{dt}$ equations

$$\cancel{\int \hat{x} \frac{d^3q}{(2\pi)^3} \frac{q}{E}} \qquad \cancel{\int \hat{x} \frac{d^3Q}{(2\pi)^3} \frac{Q}{E}}$$

$$\int \frac{d^3q}{(2\pi)^3}\, \bar{q}\, \frac{df_e}{dt}(\bar{x},\bar{q},t) = \langle C_{ep}\, \bar{q} \rangle_{QQ'q'q} + \langle C_{e\gamma}\, \bar{q} \rangle_{pp'q'q}$$

$$\int \frac{d^3\bar{Q}}{(2\pi)^3}\, \bar{Q}\, \frac{df_p}{dt}(\bar{x},\bar{q},t) = \langle C_{ep}\, \bar{Q} \rangle_{\cancel{QQ'q}\; qq'Q'Q}$$

Recall for DM we did the same but with $\int \frac{d^3p}{(2\pi)^3} \frac{p\hat{p}^i}{E} = \int \frac{d^3p}{(2\pi)^3} \frac{\bar{p}}{E}$

So here we can borrow the same results up to a factor of $E \to m_e, m_p$

For DM we got

$$\frac{\partial (n_{dm} v^j)}{\partial t} + 4H\, n_{dm} v^j + \frac{n_{dm}}{a} \frac{\partial \psi}{\partial x^j} = 0$$

Here we get:

$$m_e \frac{\partial (n_b v_b^j)}{\partial t} + 4H\, n_b v_b^j m_e + \frac{n_b}{a} m_e \frac{\partial \psi}{\partial x^j} = \langle C_{ep}\, \bar{q} \rangle_{QQ'q'q} + \langle C_{e\gamma}\, \bar{q} \rangle_{pp'q'q}$$

$$+\; m_p \frac{\partial (n_b v_b^j)}{\partial t} + 4H\, m_p n_b v_b^j + m_p \frac{n_b}{a} \frac{\partial \psi}{\partial x^j} = \langle C_{ep}\, \bar{Q} \rangle_{QQ'q'q}$$

$$(\cancel{m_e} + m_p)\left[\frac{\partial (n_b v^j)}{\partial t} + 4H\, n_b v_b^j + \frac{n_b}{a} \frac{\partial \psi}{\partial x^j} \right] = \cancel{\langle C_{ep}(q^j + Q^j) \rangle_{QQ'q'q}} + \langle C_{e\gamma}\, q^j \rangle_{pp'q'q}$$

$m_p \gg m_e$

0 again mathematically since $q^j + Q^j$ symmetric AND conserved!!!

otherwise get 2nd order terms

$n_b \to n_b^{(0)} \propto a^{-3}$ so $\frac{\partial}{\partial t} n_b^{(0)} = -3H n_b^{(0)}$

$$m_p \frac{\partial}{\partial t} n_b^{(0)} v^j + m_p n_b^{(0)} \frac{\partial v^j}{\partial t} + 4H m_p n_b^{(0)} v_b^j + m_p \frac{n_b^{(0)}}{a} \frac{\partial \psi}{\partial x^j} = \langle C_{e\gamma}\, q^j \rangle_{pp'q'q}$$

$$\cancel{-3H m_p n_b^{(0)} v^j} + m_p n_b^{(0)} \frac{\partial v^j}{\partial t} \cancel{+4} H m_p n_b^{(0)} v_b^j + m_p \frac{n_b^{(0)}}{a} \frac{\partial \psi}{\partial x^j} = \langle C_{e\gamma}\, q^j \rangle_{pp'q'q}$$

$$\rightarrow \frac{m_p n_b^{(0)}}{\rho_b}\frac{\partial v_b^J}{\partial t} + \frac{H m_p n_b^{(0)} v_b^J}{\rho_b} + \frac{m_p n_b^{(0)}}{a\,\rho_b}\frac{\partial \Psi}{\partial x^J} = \frac{\langle C_{e\gamma}\, q^J \rangle_{pp'q'q}}{\rho_b}$$

$$\Big\downarrow \ \rho_b \simeq m_p n_b^{(0)}$$

$$\frac{\partial v_b^J}{\partial t} + H v_b^J + \frac{1}{a}\frac{\partial \Psi}{\partial x^J} = \frac{1}{\rho_b}\langle C_{e\gamma}\, q^J \rangle_{pp'q'q}$$

$$\langle C_{e\gamma}\,\vec{q}\,\rangle_{pp'q'q} = -\langle C_{e\gamma}\,\vec{p}\,\rangle_{pp'q'q}$$ since $\langle C_{e\gamma}(\vec{q}+\vec{p})\rangle_{pp'q'q} = 0$

by the conservation of momentum

$$-\frac{\langle C_{e\gamma}\,\vec{p}\,\rangle_{pp'q'q}}{\rho_b}$$ → go to Fourier space

→ multiply by $\hat{k}_J$ (similarly for LHS)

$\hat{k}\cdot\vec{p} = p\mu$ →

$$-\frac{\langle C_{e\gamma}\, p\mu \rangle_{pp'q'q}}{\rho_b}$$

Recall we already computed $\langle C_{e\gamma}\rangle_{p'q'q}$ for the photon Boltzmann equation

$$\langle C_{e\gamma}\rangle_{p'q'q} = -p\frac{\partial f^{(0)}}{\partial p} n_e\sigma_T\left[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{v}_b\right]$$

$$-\frac{\langle C_{e\gamma}\, p\mu\rangle_{pp'q'q}}{\rho_b} \overset{\text{Fourier space}}{=} \frac{n_e\sigma_T}{\rho_b}\int \frac{d^3p}{(2\pi)^3}\, p^2\frac{\partial f^{(0)}}{\partial p}\,\mu\left[\tilde{\Theta}_0 - \tilde{\Theta}(\mu) + \tilde{v}_b\mu\right] =$$

$$= \frac{n_e\sigma_T}{\rho_b}\int_0^\infty \frac{dp}{2\pi^2}\, p^4\frac{\partial f^{(0)}}{\partial p}\int_{-1}^{1}\frac{d\mu}{2}\,\mu\left[\tilde{\Theta}_0 - \tilde{\Theta}(\mu) + \tilde{v}_b\mu\right]$$

$$\int\frac{d^3p}{(2\pi)^3} = \int\frac{dp\,p^2}{(2\pi)^3}\int d\varphi\int d(\cos\theta) = \int\frac{dp\,p^2}{8\pi^3}\,2\pi\int d\mu =$$

$$= \int\frac{dp\,p^2}{4\pi^2}\int d\mu = \int\frac{dp\,p^2}{2\pi^2}\int\frac{d\mu}{2}$$

$$\frac{n_e\sigma_T}{\rho_b}\underbrace{\int_0^\infty \frac{dp}{2\pi^2} p^4 \frac{\partial f^{(0)}}{\partial p}}\int_{-1}^{1}\frac{d\mu}{2}\mu\left[\tilde{\Theta}_0-\tilde{\Theta}(\mu)+\tilde{v}_b\mu\right]$$

$$\Rightarrow \int_0^\infty \frac{dp}{2\pi^2}p^4\frac{\partial f^{(0)}}{\partial p} = -4\int_0^\infty \frac{dp}{2\pi^2}p^3 f^{(0)} = -4\int_0^\infty \frac{dp\, p^2}{2\pi^2}p\, f^{(0)} = -4\int\frac{d^3p}{(2\pi)^3}p\, f^{(0)} = -4\rho_\gamma$$

$$\int_{-1}^{1}\frac{d\mu}{2}\mu\left[\cancel{\tilde{\Theta}_0}-\tilde{\Theta}(\mu)+\underbrace{\tilde{v}_b\mu}\right] = i\tilde{\Theta}_1+\frac{\tilde{v}_b}{3}$$

0 (odd)

$$\int_{-1}^{1}\frac{d\mu}{2}\tilde{v}_b\mu^2 = \tilde{v}_b\frac{\mu^3}{6}\Big|_{-1}^{1} = \tilde{v}_b\left(\frac{1}{6}+\frac{1}{6}\right) = \frac{\tilde{v}_b}{3}$$

Recall $\Theta_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)$ $\quad P_1(\mu)=\mu$

so $i\int_{-1}^{1}\frac{d\mu}{2}\mu\tilde{\Theta}(\mu) = \tilde{\Theta}_1 \rightarrow -\int\frac{d\mu}{2}\mu\tilde{\Theta}(\mu) = i\tilde{\Theta}_1$

$$\longrightarrow = -4\frac{n_e\sigma_T}{\rho_b}\rho_\gamma\left(i\tilde{\Theta}_1+\frac{\tilde{v}_b}{3}\right)$$

~~Putting everything together~~

$$\cancel{\frac{\partial v_b^j}{\partial t}+\frac{da/dt}{a}v_b^j+\frac{1}{a}\frac{\partial\Psi}{\partial x^j} = -\frac{n_e\sigma_T}{\rho_b}\rho_\gamma\left(i\Theta_1 \ldots\right)}$$

Putting everything together

LHS: $\frac{\partial v_b^j}{\partial t}+\frac{da/dt}{a}v_b^j+\frac{1}{a}\frac{\partial\Psi}{\partial x^j} \xrightarrow{t\to\eta} \frac{1}{a}\dot{v}_b+\frac{\dot{a}}{a^2}v_b^j+\frac{1}{a}\frac{\partial\Psi}{\partial x^j} \xrightarrow{FT} \frac{1}{a}\dot{\tilde{v}}_b+\frac{\dot{a}}{a^2}\tilde{v}_b+\frac{ik\tilde{\Psi}}{a}$

RHS: $-4n_e\sigma_T\frac{\rho_\gamma}{\rho_b}\left(i\tilde{\Theta}_1+\frac{\tilde{v}_b}{3}\right)$ already in Fourier space!

$$\longrightarrow \frac{1}{a}\dot{\tilde{v}}_b+\underbrace{\frac{\dot{a}}{a^2}}_{\mathcal{H}}\tilde{v}_b+\frac{ik\tilde{\Psi}}{a} = -4n_e\sigma_T\frac{\rho_\gamma}{\rho_b}\left(i\tilde{\Theta}_1+\frac{\tilde{v}_b}{3}\right)$$

$$\Rightarrow \dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + ik\tilde{\Psi} = -4 n_e \sigma_T a \frac{\rho_\gamma}{\rho_b}\left(i\tilde{\Theta}_1 + \frac{\tilde{v}_b}{3}\right)$$

Recall definition of optical depth

$$\tau(\eta) \equiv \int_\eta^{\eta_0} d\eta'\, n_e \sigma_T a \longrightarrow \dot{\tau} \equiv \frac{d\tau}{d\eta} = -n_e \sigma_T a$$

$$\boxed{\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b + ik\tilde{\Psi} = \dot{\tau}\frac{4\rho_\gamma}{3\rho_b}\left[3i\tilde{\Theta}_1 + \tilde{v}_b\right]}$$ *

Boltzmann equation for baryon velocity (1st order/1st moment)

$\tilde{\Theta}_1$ "~" v_γ so $[\dots] \sim v_\gamma - v_b$: momentum exchange

Physical meaning of $\frac{1}{\rho_b}$ factor : moving e^- is difficult because they are tightly coupled to heavy protons. If $m_p \to \infty$, $\rho_b \to \infty$, Compton scattering cannot change electron velocity

$$R \equiv \frac{3\rho_b}{4\rho_\gamma} = \frac{(1+w_b)\rho_b}{(1+w_\gamma)\rho_\gamma}$$ ~~pc~~ baryon-to-photon density ratio

* even if there is neutral hydrogen so $n_e \neq n_b$, approximately valid

Recap Boltzmann equations so far

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau}\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b\right]$$

$$\dot{\tilde{\delta}}_{dm} + ik\tilde{v}_{dm} = -3\dot{\tilde{\Phi}}$$

$$\dot{\tilde{v}}_{dm} + \frac{\dot{a}}{a}\tilde{v}_{dm} = -ik\tilde{\Psi}$$

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_b = -3\dot{\tilde{\Phi}}$$

$$\dot{\tilde{v}}_b + \frac{\dot{a}}{a}\tilde{v}_b = -ik\tilde{\Psi} + \frac{\dot{\tau}}{R}\left[\tilde{v}_b + 3i\tilde{\Theta}_1\right]$$

Summary of Boltzmann equations for γ, DM, b, ν

+ other effects we glossed over

(Drop ~ from Fourier transforms)

massless ν basically like γ, just not interacting! $\to$ no scattering

$$\boxed{\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik_\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]}$$

$\Pi = \Theta_2 + \Theta_{P2} + \Theta_{P0}$ temperature field also coupled + polarization field

angular dependence of Compton scattering $|M|^2 \propto |\hat{\varepsilon}\cdot\hat{\varepsilon}'|^2$

$$\dot{\Theta}_P + ik_\mu\Theta_P = -\dot{\tau}\left[-\Theta_P + \frac{1}{2}\left(1 - P_2(\mu)\right)\Pi\right]$$ evolution of polarization field

$$\dot{\delta}_{dm} + ikv_{dm} = -3\dot{\Phi}$$

$$\dot{v} + \frac{\dot{a}}{a}v = -ik\Psi$$

$$\dot{\delta}_b + ikv_b = -3\dot{\Phi}$$

$$\dot{v}_b + \frac{\dot{a}}{a}v_b = -ik\Psi + \frac{\dot{\tau}}{R}\left[v_b + 3i\Theta_1\right]$$

$$\dot{N} + ik_\mu N = -\dot{\Phi} - ik_\mu\Psi$$

These equations tell us how particle distributions are affected by metric fluctuations. Now we need to know the other way around $\to$ equations for Φ, Ψ!

$\downarrow$

(perturbed) Einstein equations

massless neutrinos, without scattering, so equation identical to that for photons, minus scattering

$\to$ typically expanded in multipoles

$$\Theta_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)$$

e.g. first two moments neglecting scattering

$$\begin{cases} \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \\ \dot{\Theta}_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\Phi \end{cases}$$