THE (PERTURBED) EINSTEIN EQUATIONS

particle distributions $\xrightarrow{\text{Einstein equations}}$ perturbations in the gravitational field

$f \Rightarrow \boxed{\Theta, \delta, v, N}$ $\xleftarrow{\text{Boltzmann equations}}$ $\boxed{\Phi, \Psi}$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$

expand perturbatively around zero-order homogeneous solution

LHS 3 steps:

1) $\Gamma^{\mu}_{\alpha\beta}$ for perturbed metric $ds^2 = -(1+2\Psi)dt^2 + a^2(1+2\Phi)\delta_{ij}dx^i dx^j$

2) $R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$

3) $R = g^{\mu\nu} R_{\mu\nu}$

Perturbed Ricci tensor and scalar (for scalar perturbations)

Christoffel symbols

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]$$

0th order

$\Gamma^0_{00} = 0$ $\quad \Gamma^0_{0i} = \Gamma^0_{i0} = 0$ $\quad \Gamma^0_{ij} = \delta_{ij}\dot{a}a$ $\quad \Gamma^i_{0j} = \Gamma^i_{j0} = \delta_{ij}\frac{\dot{a}}{a}$ $\quad$ all other Γ's $= 0$

1st order: only take terms linear in Φ or Ψ

Start from $\Gamma^0_{\mu\nu}$

$$\Gamma^0_{\mu\nu} = \frac{g^{0\alpha}}{2}\left[\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right]$$

$$\Gamma^0_{\mu\nu} = \frac{g^{0\alpha}}{2}\left[\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right]$$

$\alpha = 0$

$$g^{00} = (g_{00})^{-1} = \cancel{(-1-2\psi)}\left[(1+2\psi)\right]^{-1} \simeq -1+2\psi$$

$$\Gamma^0_{\mu\nu} \simeq \frac{-1+2\psi}{2}\left[\frac{\partial g_{0\mu}}{\partial x^\nu} + \frac{\partial g_{0\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial t}\right]$$

Look at components

- $\mu=0,\ \nu=0$ $\quad \Gamma^0_{00} \simeq -\frac{1+2\psi}{2}\left[\frac{\partial g_{00}}{\partial t} + \cancel{\frac{\partial g_{00}}{\partial t}} - \cancel{\frac{\partial g_{00}}{\partial t}}\right] \simeq \left(\frac{-1+2\psi}{2}\right)\frac{\partial g_{00}}{\partial t}$

$$g_{00} = -1-2\psi \rightarrow \frac{\partial g_{00}}{\partial t} = -2\frac{\partial \psi}{\partial t} \longrightarrow \Gamma^0_{00} \simeq \left(\frac{-1+2\psi}{2}\right) \times -2\frac{\partial \psi}{\partial t}$$

$$\rightarrow \Gamma^0_{00} \simeq \frac{\partial \psi}{\partial t}$$

drop 2nd order term $\propto \psi\frac{\partial \psi}{\partial t}$

- $\mu=0,\ \nu=i$ or $\mu=i,\ \nu=0$ (Γ symmetric)

$\cancel{\Gamma^0_{0i} = \frac{g^{0\alpha}}{2}\left[\frac{\partial g}{\partial}\right.}$ $\quad \Gamma^0_{0i} \simeq \frac{-1+2\psi}{2}\left[\frac{\partial g_{00}}{\partial x^i} + \underbrace{\cancel{\frac{\partial g_{0i}}{\partial t}}}_{0} - \underbrace{\cancel{\frac{\partial g_{0i}}{\partial t}}}_{0}\right]$

$$= \left(\frac{-1+2\psi}{2}\right)\frac{\partial g_{00}}{\partial x^i} = \left(\frac{-1+2\psi}{2}\right) \times -2\frac{\partial \psi}{\partial x^i} \simeq \frac{\partial \psi}{\partial x^i} \longrightarrow ik_i\psi$$

$\frac{\partial g_{00}}{\partial x^i} = -2\frac{\partial \psi}{\partial x^i}$

drop 2nd order term $\propto \psi\frac{\partial \psi}{\partial x^i}$

Fourier space

- $\mu=i,\ \nu=j$

$$\Gamma^0_{ij} \simeq \frac{-1+2\psi}{2}\left[\underbrace{\cancel{\frac{\partial g_{0i}}{\partial x^j}}}_{0} + \underbrace{\cancel{\frac{\partial g_{0j}}{\partial x^i}}}_{0} - \frac{\partial g_{ij}}{\partial t}\right] = \frac{1-2\psi}{2}\frac{\partial}{\partial t}\left[\delta_{ij}a^2(1+2\Phi)\right]$$

$$\frac{\partial}{\partial t}\left[\delta_{ij}a^2(1+2\Phi)\right] = \delta_{ij}2a\frac{da}{dt}(1+2\Phi) + 2\delta_{ij}a^2\frac{\partial\Phi}{\partial t} = \delta_{ij}a^2\left(2\frac{1}{a}\frac{da}{dt}(1+2\Phi) + 2\frac{\partial\Phi}{\partial t}\right) =$$

$$=\cancel{\delta_{ij}a^2\left(2H(1+2\Phi)+2\frac{\partial\Phi}{\partial t}\right)} \qquad = \delta_{ij}a^2\left(2H(1+2\Phi)+2\frac{\partial\Phi}{\partial t}\right)$$

$$\Gamma^0_{ij} \simeq \frac{1-2\Psi}{2}\frac{\partial}{\partial t}\left[\delta_{ij}a^2(1+2\Phi)\right] = \frac{(1-2\Psi)}{2}\delta_{ij}a^2\left(2H(1+2\Phi)+2\frac{\partial\Phi}{\partial t}\right) =$$

2nd order   2nd order

$$= \delta_{ij}\frac{a^2}{2}\left[2H+4H\Phi+2\frac{\partial\Phi}{\partial t}-4H\Psi-\cancel{4H4\Psi\Phi}-\cancel{4\Psi 2\frac{\partial\Phi}{\partial t}}\right]=$$

$$= \delta_{ij}\frac{a^2}{2}\left[2H+4H(\Phi-\Psi)+2\frac{\partial\Phi}{\partial t}\right] = \delta_{ij}a^2\left[H+2H(\Phi-\Psi)+\frac{\partial\Phi}{\partial t}\right]$$

Other Christoffel symbols, exercise

In summary

$$\Gamma^0_{00} = \frac{\partial\Psi}{\partial t} \qquad \Gamma^0_{0i} = \Gamma^0_{i0} = ik_i\Psi \qquad \Gamma^0_{ij} = \delta_{ij}a^2\left[H+2H(\Phi-\Psi)+\frac{\partial\Phi}{\partial t}\right]$$

$$\Gamma^i_{00} = \frac{ik^i}{a^2}\Psi \qquad \Gamma^i_{j0} = \Gamma^i_{0j} = \delta_{ij}\left(H+\frac{\partial\Phi}{\partial t}\right)$$

only non-vanishing zero-order components

$$\Gamma^i_{jk} = i\Phi\left(\delta_{ij}k_k+\delta_{ik}k_j-\delta_{jk}k_i\right)$$

note: $\delta_{ij} = \delta^{ij}$ $k_i = k^i$

Ricci tensor

$$R_{00} = \partial_\alpha\Gamma^\alpha_{00} - \partial_0\Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha}$$

consider $\alpha=0$ $\cancel{\partial_0\Gamma^0_{00}}-\cancel{\partial_0\Gamma^0_{0}}+\cancel{\Gamma^0_{\beta\alpha}\Gamma^\beta_{00}}-\cancel{\Gamma^0_{\beta 0}\Gamma^\beta_{00}}$

$\hookrightarrow$ so sum over index α only contributes when $\alpha = i$

$$R_{00} = \underbrace{\partial_i\Gamma^i_{00}}_{①} - \underbrace{\partial_0\Gamma^i_{0i}}_{②} + \underbrace{\Gamma^i_{\beta i}\Gamma^\beta_{00}}_{③} - \underbrace{\Gamma^i_{\beta 0}\Gamma^\beta_{0i}}_{④}$$

① $\partial_i\Gamma^i_{00} \longrightarrow ik_i\Gamma^i_{00} = ik_i\frac{ik^i}{a^2}\Psi = -\frac{k^2}{a^2}\Psi$

Fourier space

$$\partial_i\Gamma^i_{00} = -\frac{k^2}{a^2}\Psi$$

② $-\partial_0 \Gamma^i_{\ 0i} = -\dfrac{\partial}{\partial t}\left[\underbrace{\delta_{ij}}_{\text{"}\times 3\text{"}}\left(H + \dfrac{\partial \Phi}{\partial t}\right)\right] = -3\dfrac{\partial}{\partial t}\left(H + \dfrac{\partial \Phi}{\partial t}\right)$ *

$$\frac{dH}{dt} = \frac{d}{dt}\left(\frac{1}{a}\frac{da}{dt}\right) = \cancel{\frac{1}{a}\frac{d^2a}{dt^2}} = \frac{a\, d^2a/dt^2 - (da/dt)^2}{a^2} = \frac{1}{a}\frac{d^2a}{dt^2} - \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{1}{a}\frac{d^2a}{dt^2} - H^2$$

$$* = -3\left(\frac{d^2a/dt^2}{a} - H^2 + \frac{\partial^2 \Phi}{\partial t^2}\right)$$

③ $\underbrace{\Gamma^i_{\ i\beta}}_{\text{only keep 0th order part}}\underbrace{\Gamma^\beta_{\ 00}}_{\text{1st order}}$ $\quad \Gamma^\beta_{\ 00} = \dfrac{ik^\beta}{a^2}\Psi \ (\beta = i), \ \dfrac{\partial \Psi}{\partial t} \ (\beta = 0) \longrightarrow$ 1st order!

only keep 0th order part $\to$

$\Gamma^i_{\ i\beta} \sim i\Phi\left[\delta K + \delta K - \delta K\right]$ always 1st order

$\Gamma^i_{\ i0} = \delta_{ii}\left(H + \dfrac{\partial \Phi}{\partial t}\right)$ has 0th order part

need $\beta = 0$!

$$\Rightarrow \Gamma^i_{\ i\beta}\Gamma^\beta_{\ 00} \simeq \cancel{\delta_{ii}H}\ \Gamma^i_{\ i0}\Gamma^0_{\ 00} \simeq \delta_{ii} H \frac{\partial \Psi}{\partial t} = 3H\frac{\partial \Psi}{\partial t}$$

$\Gamma^i_{\ i\beta}$ ~~0th~~ 0th-order part, $\beta = 0$

δ_{ii} = "$\times 3$"

④ $-\Gamma^i_{\ \beta 0}\Gamma^\beta_{\ 0i}$ $\quad \beta = 0 \sim \underbrace{\Gamma^i_{\ 00}}_{\substack{\text{1st order} \\ \frac{ik^i\Psi}{a^2}}}\underbrace{\Gamma^0_{\ 0i}}_{\substack{\text{1st order} \\ ik^i\Psi}} \sim$ 2nd order $\to$ must be $\beta = j$!

$\beta = j \to$

$$\simeq -\Gamma^i_{\ j0}\Gamma^j_{\ 0i} = -\delta_{ij}\left(H + \frac{\partial \Phi}{\partial t}\right)\delta_{ji}\left(H + \frac{\partial \Phi}{\partial t}\right) =$$

$\delta_{ij}\delta_{ji} = \delta_{ij}$

$$= -\delta_{ij}\left(H + \frac{\partial \Phi}{\partial t}\right)\left(H + \frac{\partial \Phi}{\partial t}\right) \simeq -3\left(H^2 + 2H\frac{\partial \Phi}{\partial t}\right)$$

δ_{ij} = "$\times 3$"

Putting everything together

$$R_{00} = ① + ② + ③ + ④ = -\frac{k^2}{a^2}\Psi - 3\left(\frac{d^2a/dt^2}{a} - H^2 + \frac{\partial^2 \Phi}{\partial t^2}\right) + 3H\frac{\partial \Psi}{\partial t} - 3\left(H^2 + 2H\frac{\partial \Phi}{\partial t}\right) =$$

$$= -\frac{k^2}{a^2}\Psi - 3\frac{d^2a/dt^2}{a} \cancel{+3H^2} - 3\frac{\partial^2 \Phi}{\partial t^2} + 3H\frac{\partial \Psi}{\partial t} \cancel{-3H^2} - 6H\frac{\partial \Phi}{\partial t} =$$

$$\Rightarrow \boxed{\begin{aligned} R_{00} &= -3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\psi - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\left(\frac{\partial\psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right) \\ &\text{(exercise)} \\ R_{ij} &= \delta_{ij}\left[\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right)(1+2\Phi-2\psi) + a^2H\left(6\frac{\partial\Phi}{\partial t} - \frac{\partial\psi}{\partial t}\right) + a^2\frac{\partial^2\Phi}{\partial t^2} + k^2\Phi\right] \\ &\quad + k_ik_j(\Phi+\psi) \end{aligned}}$$

<u>Ricci scalar</u>

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij} =$$

$$g^{00} \simeq \left[-(1+2\psi)\right]^{-1} \simeq -1+2\psi$$

$$g^{ij} = \left[\delta_{ij}a^2(1+2\Phi)\right]^{-1} \simeq \frac{\delta_{ij}}{a^2}(1-2\Phi)$$

$$= \underbrace{R_0}_{\text{0th order}} + \underbrace{\delta R}_{\text{1st order}}$$

$$R = g^{00}R_{00} + g^{ij}R_{ij} = (-1+2\psi)\left[-3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\psi - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\left(\frac{\partial\psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right)\right]$$

$$+ \delta_{ii}\left(\frac{1-2\Phi}{a^2}\right)\left[\delta_{ii}\, 3\left\{\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right)(1-2\Phi-2\psi) + a^2H\left(6\frac{\partial\Phi}{\partial t} - \frac{\partial\psi}{\partial t}\right) + a^2\frac{\partial^2\Phi}{\partial t^2} + k^2\Phi\right\}\right.$$

$$\left. + k^2(\Phi+\psi)\right]$$

Look at 0th order pieces

$$R_0 = -1 \times -3\frac{1}{a}\frac{d^2a}{dt^2} + \frac{1}{a^2}\times 3\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right) = 3\frac{1}{a}\frac{d^2a}{dt^2} + 6H^2 + 3\frac{1}{a}\frac{d^2a}{dt^2} =$$

$$= 6\left(H^2 + \frac{d^2a/dt^2}{a}\right) \checkmark \longrightarrow$$ already seen in 2nd Friedmann equation!

$$\delta R = \left(g^{00}R_{00}\right)_{1st} + \left(g^{ij}R_{ij}\right)_{1st}$$

$$\left(g^{00}R_{00}\right)_{1st} = \overbrace{\cancel{3\frac{d^2a}{dt^2}}}^{\text{0th}} + \frac{k^2}{a^2}\psi + 3\frac{\partial^2\Phi}{\partial t^2} - 3H\frac{\partial\psi}{\partial t} + 6H\frac{\partial\Phi}{\partial t} - 6\psi\frac{d^2a/dt^2}{a} - \overbrace{\cancel{2\frac{k^2}{a^2}\psi^2}}^{\text{2nd}} - \overbrace{\cancel{6\psi\frac{\partial^2\Phi}{\partial t^2}}}^{\text{2nd}}$$

$$+ \overbrace{\cancel{6H\psi\frac{\partial\psi}{\partial t}}}^{\text{2nd}} - \overbrace{\cancel{12H\psi\frac{\partial\Phi}{\partial t}}}^{\text{2nd}} \simeq -6\psi\frac{d^2a/dt^2}{a} + \frac{k^2}{a^2}\psi + 3\frac{\partial^2\Phi}{\partial t^2} - 3H\left(\frac{\partial\psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right)$$

$$(g^{ij}R_{ij})_{1st} = \left(\frac{1-2\Phi}{a^2}\right)\Big[6a^2H^2 + 12a^2H^2\Psi - 12a^2H^2\Psi + 3a\frac{d^2a}{dt^2} + 6a\frac{d^2a}{dt^2}\Phi - 6a\frac{d^2a}{dt^2}\Psi$$

$$+ 18a^2H\frac{\partial\Phi}{\partial t} - 3a^2H\frac{\partial\Psi}{\partial t} + 3a^2\frac{\partial^2\Phi}{\partial t^2} + 3k^2\Phi + k^2\Phi + k^2\Psi\Big] =$$

$$= \left(\frac{1-2\Phi}{a^2}\right)\Big[-6\Psi\left(a\frac{d^2a}{dt^2} + 2a^2H^2\right) + 3H\left(6a^2\frac{\partial\Phi}{\partial t} - a^2\frac{\partial\Psi}{\partial t}\right)$$

$$+ 3a\frac{d^2a}{dt^2}(1+2\Phi) + 6a^2H^2(1+2\Phi)$$

$$+ 3a^2\frac{\partial^2\Phi}{\partial t^2} + 4k^2\Phi + k^2\Psi\Big] \simeq$$

$(1+2\Phi)(1-2\Phi) \simeq 1$, left with ~~$3a\frac{d^2a}{dt^2} + 6a^2H^2$, both 0th order~~

all 1st order, multiply by $\frac{1}{a^2}$

$$\simeq -6\Psi\left(2H^2 + \frac{d^2a/dt^2}{a}\right) + 3H\left(6\frac{\partial\Phi}{\partial t} - \frac{\partial\Psi}{\partial t}\right) + 3\frac{\partial^2\Phi}{\partial t^2} + 4\frac{k^2}{a^2}\Phi + \frac{k^2}{a^2}\Psi$$

Putting everything together...

$$\delta R = -6\Psi\frac{d^2a/dt^2}{a} + \frac{k^2}{a^2}\Psi + 3\frac{\partial^2\Phi}{\partial t^2} - 3H\left(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right) - 6\Psi\left(2H^2 + \frac{d^2a/dt^2}{a}\right)$$

$$+ 3H\left(6\frac{\partial\Phi}{\partial t} - \frac{\partial\Psi}{\partial t}\right) + 3\frac{\partial^2\Phi}{\partial t^2} + 4\frac{k^2}{a^2}\Phi + \frac{k^2}{a^2}\Psi =$$

$$= -12\Psi\left(H^2 + \frac{d^2a/dt^2}{a}\right) + \frac{2k^2}{a^2}\Psi + 6\frac{\partial^2\Phi}{\partial t^2} - 6H\left(\frac{\partial\Psi}{\partial t} - 4\frac{\partial\Phi}{\partial t}\right) + 4\frac{k^2}{a^2}\Phi$$

$$\boxed{\delta R = -12\Psi\left(H^2 + \frac{d^2a/dt^2}{a}\right) + \frac{2k^2}{a^2}\Psi + 6\frac{\partial^2\Phi}{\partial t^2} - 6H\left(\frac{\partial\Psi}{\partial t} - 4\frac{\partial\Phi}{\partial t}\right) + 4\frac{k^2}{a^2}\Phi}$$

Two components of the Einstein equations

Now derive evolution equations for Φ, Ψ

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$\hookrightarrow$ 10 components, but we need only 2! (for scalar)

So let's look only at 2 components:

- time-time (00)
- longitudinal, traceless part of space-space ($\sim \hat{k}_i \hat{k}_j$ ij)

$$G^0{}_0 = g^{00}\left[R_{00} - \frac{1}{2} g_{00} R\right] = g^{00} R_{00} - \frac{1}{2} \underbrace{g^{00} g_{00}}_{=1} R \simeq (-1+2\Psi) R_{00} - \frac{R}{2}$$

$\hookrightarrow$ in hindsight simplifies $T^\mu{}_\nu$

Already computed R_{00} ~~earlier~~ and δR earlier

$$R_{00} = -3\frac{d^2a/dt^2}{a} - \frac{k^2}{a^2}\Psi - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\left(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right)$$

$$(\delta R) = -12\Psi\left(H^2 + \frac{d^2a/dt^2}{a}\right) + \frac{2k^2}{a^2}\Psi + 6\frac{\partial^2\Phi}{\partial t^2} - 6H\left(\frac{\partial\Psi}{\partial t} - 4\frac{\partial\Phi}{\partial t}\right) + \frac{4k^2\Phi}{a^2}$$

1st order

$$\Rightarrow (\delta G^0{}_0)_{1st} = \cancel{(\delta g_{00} \ldots)} \left[(-1+2\Psi)R_{00}\right]_{1st} - \frac{(\delta R)_{1st}}{2}$$

$$\simeq \underbrace{2\Psi \times \cancel{-3\frac{d^2a/dt^2}{a}}}_{\times 2\Psi} \underbrace{+\cancel{\frac{k^2}{a^2}\Psi} + \cancel{3\frac{\partial^2\Phi}{\partial t^2}} - 3H\left(\cancel{\frac{\partial\Psi}{\partial t}} - 2\frac{\partial\Phi}{\partial t}\right)}_{\times -1 \text{ since } \Psi \text{ already is 1st order}} + 6\Psi\left(H^2 + \cancel{\frac{d^2a/dt^2}{a}}\right) - \cancel{\frac{k^2}{a^2}\Psi}$$

$$\cancel{-3\frac{\partial^2\Phi}{\partial t^2}} + 3H\left(\cancel{\frac{\partial\Psi}{\partial t}} - 4\frac{\partial\Phi}{\partial t}\right) - \frac{2k^2\Phi}{a^2} =$$

$$= -6H\frac{\partial\Phi}{\partial t} + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2}$$

$$\rightarrow \boxed{(\delta G^0{}_0)_{1st} \simeq -6H\frac{\partial\Phi}{\partial t} + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2}} \quad \left[= 8\pi G (T^0{}_0)_{1st}\right]$$

Now need 1st-order part of $T^0{}_0$

Recall: $-T^0{}_0$ energy density of all particles ($\sim \int f$)

$\hookrightarrow T^\mu{}_\nu = \mathrm{diag}(-\rho, P, P, P)$

so

$$T^0{}_0(\bar{x},t) = -\sum_i g_i \int \frac{d^3p}{(2\pi)^3} E_i(p)\, \underbrace{f_i(\bar{p},\bar{x},t)}$$

all species

look at 1st-order part for γ, ν, DM, b

DM, baryons:

By definition $\quad T^0{}_0(\bar{x},t) = -\sum_i \bar{\rho}_i(1+\delta_i) \quad i = \mathrm{DM}, b$

$\hookrightarrow \delta T^0{}_0 = -\rho_i \delta_i$

Photons:

Recall definition of Θ

$$f = \left[\exp\left\{\frac{p}{T(t)[1+\Theta(\bar{x},\hat{p},t)]}\right\} - 1\right]^{-1} \underset{\Theta \ll 1}{\simeq} f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta$$

$$\rightarrow \left[T^0{}_0(\bar{x},t)\right]_\gamma = \underbrace{-2}_{g_\gamma} \int \frac{d^3p}{(2\pi)^3} p\left[\underset{1}{f^{(0)}} - \underset{2}{p\frac{\partial f^{(0)}}{\partial p}\Theta}\right]$$

1 $\quad -2\int \frac{d^3p}{(2\pi)^3} p f^{(0)} = -\rho_\gamma$

2 $\quad 2\int \frac{d^3p}{(2\pi)^3} p^2 \frac{\partial f^{(0)}}{\partial p}\Theta =$

Recall $\quad \Theta_0 = \frac{1}{4\pi}\int d\Omega' \,\Theta(\hat{p}',\bar{x},t)$

$$= 2\underbrace{\int d\Omega\, \Theta}_{4\pi\Theta_0} \int \frac{dp\, p^4}{(2\pi)^3}\frac{\partial f^{(0)}}{\partial p} \underset{\text{by parts}}{=} \Theta_0\, 4\pi\left(-2\int \frac{dp\, 4p^3 f^{(0)}}{(2\pi)^3}\right) =$$

$$= -4\Theta_0\left[2\int \frac{dp\, p^3 4\pi f^{(0)}}{(2\pi)^3}\right] = -4\rho_\gamma\Theta_0 .$$

$\hookrightarrow 2\int \frac{d^3p}{(2\pi)^3} p f^{(0)} = \rho_\gamma$

$$\boxed{(T^0{}_0)_{\gamma} = -\rho_\gamma [1 + 4\Theta_0]} \qquad \delta T^0{}_0 = -4\rho_\gamma \Theta_0$$

factor of 4 makes sense in retrospect

Stefan-Boltzmann law $\rho_\gamma \propto T^4$, $\Theta \equiv \frac{\delta T}{T}$

$$\rho_\gamma \propto T^4 \Rightarrow \delta = \frac{\delta\rho_\gamma}{\rho_\gamma} \sim \frac{\delta(T^4)}{T^4} \sim \frac{4T^3\delta T}{T^4} \sim 4\frac{\delta T}{T} = 4\Theta$$

For massless neutrinos identical calculations

$$\boxed{(T^0{}_0)_\nu = -\rho_\nu [1 + 4N_0]} \qquad \delta T^0{}_0 = -4\rho_\nu N_0$$

(neglecting dark energy from sources of perturbations)

Putting everything together

$$\delta G^0{}_0 = 8\pi G\, \delta T^0{}_0$$

$$\Rightarrow -6H\frac{\partial\Phi}{\partial t} + 6\Psi H^2 - \frac{2k^2\Phi}{a^2} = -8\pi G\left[\rho_{dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0\right]$$

$$\Downarrow$$

$$-3H\frac{\partial\Phi}{\partial t} + 3\Psi H^2 - \frac{k^2\Phi}{a^2} = -4\pi G\left[\rho_{dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0\right]$$

conformal time $d\eta = \frac{dt}{a} \to \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta}$

$H = \frac{1}{a}\frac{da}{dt} = \frac{1}{a}\left(\frac{1}{a}\frac{da}{d\eta}\right) = \frac{1}{a}\frac{\dot a}{a} = \frac{\mathcal{H}}{a}$

$$-3\frac{1}{a}\frac{\dot a}{a}\frac{1}{a}\frac{\partial\Phi}{\partial\eta} + 3\Psi\frac{1}{a^2}\left(\frac{\dot a}{a}\right)^2 - \frac{k^2\Phi}{a^2} = -4\pi G\left[\rho_{dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0\right]$$

$\dot\Phi = \frac{\partial\Phi}{\partial\eta}$, $\dot\Psi = \frac{\partial\Psi}{\partial\eta}$

00 component of perturbed Einstein equation (1st order)!

$$\boxed{k^2\Phi + 3\frac{\dot a}{a}\left(\dot\Phi - \Psi\frac{\dot a}{a}\right) = 4\pi G a^2\left[\rho_{dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0\right]}$$

Without expansion ($\dot{a} \to 0$) reduces to

$$K^2 \Phi \propto 4\pi G \delta\rho \longrightarrow \underbrace{-\nabla^2 \Phi = 4\pi G \delta_\rho}_{\text{ordinary Poisson equation}}$$

Expansion terms important for modes with wavelength $\sim \frac{a}{k}$ comparable to or larger than Hubble radius

$$\text{LHS} \propto \underset{②}{K^2 \Phi} + \left(\frac{\dot{a}}{a}\right)^2 \Psi \propto k^2 \Phi + \underset{①}{\frac{a^2}{\cancel{a}} H^2 \Psi}$$

① $\gg$ ②

$\Longrightarrow$ $\Psi \sim \Phi$ $\cancel{\frac{H^2}{a^2} \gg k^2}$

$a^2 H^2 \gg k^2 \Rightarrow \frac{a}{k} \gg \frac{1}{H}$

obviously expansion important only for long-wavelength modes

This was the first perturbed Einstein equation. For the second one let's look at $G^i{}_j$

$$G^i{}_j = g^{ik}\left[R_{kj} - \frac{g_{kj}}{2} R\right] = g^{ik} R_{kj} - \frac{g^{ik} g_{kj}}{2} R \simeq \frac{\delta^{ij}(1-2\Phi)}{a^2} R_{kj} - \frac{\delta^{ij} R}{2} =$$

Recall

$$= \frac{(1-2\Phi)}{a^2} R_{ij} - \frac{\delta_{ij} R}{2}$$

$$R_{ij} \propto \delta_{ij} \underbrace{[\ldots]}_{\text{A LOT of terms}} + k_i k_j (\Phi + \Psi)$$

Includes $-\frac{\delta_{ij} R}{2}$

So $$G^i{}_j = A \underline{\delta_{ij}} + \frac{k_i k_j (\Phi + \Psi)}{a^2}$$

trace of $G^i{}_j$

To "kill" terms proportional to δ_{ij} we consider the longitudinal traceless part of $G^i{}_j$ through a projection operator

$P_i{}^j G^i{}_j$ such that result is

- longitudinal $\varepsilon_{ijk} G_{kl,jl} = 0$
- traceless $\delta^{ij} G_{ij} = 0$