$$\Rightarrow (\delta G^i{}_j)_{\substack{\text{tensor}\\ \text{1st order}}} = \delta^{ik}\left(\frac{3}{2}H\frac{\partial \mathcal{H}_{kj}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{kj}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{kj}\right)$$

From these we need two equations for $h_+, h_\times$

Recall $\mathcal{H}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \mathcal{H}_{11} = h_+ = -\mathcal{H}_{22}$

Since $\delta G^i{}_j$ involves $\mathcal{H}_{ij}, \frac{\partial \mathcal{H}_{ij}}{\partial t}, \frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2} \Longrightarrow \delta G^1{}_1 = -\delta G^2{}_2$

$$\delta G^1{}_1 = \delta^{1k}\left[\frac{3}{2}H\frac{\partial \mathcal{H}_{k1}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{k1}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{k1}\right] = \frac{3}{2}H\frac{\partial \mathcal{H}_{11}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{11}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{11}$$

$$\delta G^2{}_2 = \delta^{2k}\left[\frac{3}{2}H\frac{\partial \mathcal{H}_{k2}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{k2}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{k2}\right] = \frac{3}{2}H\frac{\partial \mathcal{H}_{22}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{22}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{22}$$

$$\Downarrow$$

$$\delta G^1{}_1 - \delta G^2{}_2 = \frac{3}{2}H\underbrace{\left(\frac{\partial \mathcal{H}_{11}}{\partial t} - \frac{\partial \mathcal{H}_{22}}{\partial t}\right)}_{2\frac{\partial \mathcal{H}_{11}}{\partial t} = 2\frac{\partial h_+}{\partial t}} + \frac{1}{2}\underbrace{\left(\frac{\partial^2 \mathcal{H}_{11}}{\partial t^2} - \frac{\partial^2 \mathcal{H}_{22}}{\partial t^2}\right)}_{2\frac{\partial^2 \mathcal{H}_{11}}{\partial t^2} = 2\frac{\partial^2 h_+}{\partial t^2}} + \frac{k^2}{2a^2}\underbrace{\left(\mathcal{H}_{11} - \mathcal{H}_{22}\right)}_{2\mathcal{H}_{11} = 2h_+} =$$

$$= 3H\frac{\partial h_+}{\partial t} + \frac{\partial^2 h_+}{\partial t^2} + \frac{k^2}{a^2}h_+$$

Change to conformal time

$$d\eta = \frac{dt}{a} \rightarrow \frac{d}{d\eta} = a\frac{d}{dt} \rightarrow \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta}$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{1}{a}\frac{d}{d\eta}\right) = \frac{1}{a}\frac{d}{d\eta}\left(\frac{1}{a}\frac{d}{d\eta}\right) = \frac{1}{a^2}\frac{d^2}{d\eta^2} - \frac{1}{a}\frac{\mathring{a}}{a^2}\frac{d}{d\eta} = \frac{1}{a^2}\frac{d^2}{d\eta^2} - \frac{\mathring{a}}{a^3}\frac{d}{d\eta}$$

$$\Rightarrow \frac{\partial h_+}{\partial t} = \frac{1}{a}\frac{dh_+}{d\eta} = \frac{\mathring{h}_+}{a} \qquad \frac{\partial^2 h_+}{\partial t^2} = \frac{1}{a^2}\frac{d^2 h_+}{d\eta^2} - \frac{\mathring{a}}{a^3}\frac{dh_+}{d\eta} = \frac{\ddot{h}_+}{a^2} - \frac{\mathring{a}}{a^3}\mathring{h}_+$$

$$\Rightarrow \delta G^1{}_1 - \delta G^2{}_2 = 3H\frac{\partial h_+}{\partial t} + \frac{\partial^2 h_+}{\partial t^2} + \frac{k^2 h_+}{a^2} = 3H\frac{\mathring{h}_+}{a} + \frac{\ddot{h}_+}{a^2} - \frac{\mathring{a}}{a^3}\mathring{h}_+ + \frac{k^2 h_+}{a^2} =$$

$$H = \frac{1}{a}\frac{da}{dt} = \frac{1}{a^2}\frac{da}{d\eta} = \frac{\mathring{a}}{a^2}$$

$$= 3\frac{\dot a}{a^2}\frac{\dot h_+}{a} + \frac{\ddot h_+}{a^2} - \frac{\dot a}{a^3}\dot h_+ + \frac{k^2 h_+}{a^2} = 2\frac{\dot a}{a^3}\dot h_+ + \frac{\ddot h_+}{a^2} + \frac{k^2 h_+}{a^2}$$

$$\Rightarrow a^2\left[\delta G^1{}_1 - \delta G^2{}_2\right] = \ddot h_+ + 2\frac{\dot a}{a}\dot h_+ + k^2 h_+$$

Easy to show that $T^1{}_1 - T^2{}_2 = 0$ if $\Theta = \Theta(\mu)$

↳ terms Θ that source scalar perturbations do not source tensor perturbations

$\rightarrow a^2\left[\delta G^1{}_1 - \delta G^2{}_2\right] = 0 \rightarrow \ddot h_+ + 2\frac{\dot a}{a}\dot h_+ + k^2 h_+ = 0$

Taking $\delta G^1{}_2$ we can follow the exact same steps for $h_\times$

$$\Rightarrow \boxed{\ddot h_\alpha + 2\frac{\dot a}{a}\dot h_\alpha + k^2 h_\alpha = 0 \qquad \alpha = +,\times}$$

Einstein equations for tensor perturbations, 1st order

→ (Damped) wave equation: gravitational waves!

If we neglect expansion of Universe

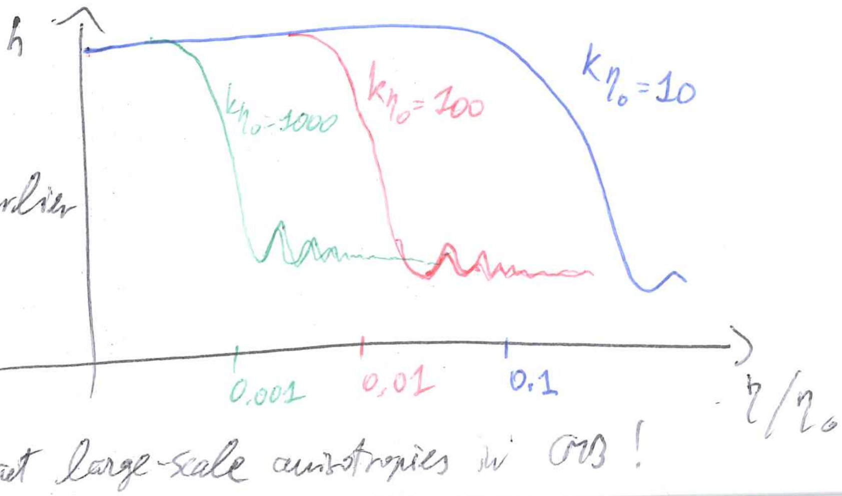
$\ddot h_\alpha + k^2 h_\alpha = 0 \rightarrow h_\alpha \propto e^{\pm ik\eta} \rightarrow h_\alpha(\bar x, \eta) = \int d^3k\, e^{i\bar k\cdot\bar x}\left[A e^{ik\eta} + B e^{-ik\eta}\right]$

GWs travelling in $\pm\hat z$ direction at speed of light

For the full equation, solutions are damped oscillations. Modes start being damped when their wavelengths enter the horizon: $k\eta \approx 1$

$k\eta \ll 1 \rightarrow h \sim \text{const}$

$k\eta \gg 1 \rightarrow h$ damped



Small-scale modes decay earlier than large-scale modes

Decoupling: $\frac{\eta}{\eta_0} \simeq 0.02$, only modes with $k\eta_0 \lesssim 100$ survive

↳ GWs only impact large-scale anisotropies in CMB!

~~Deco~~ The decomposition theorem

Perturbations to the metric: scalar, vector, tensor $\rightarrow$ each evolves independently (do not induce each other)

Recall we considered scalar perturbations from

$$G^0{}_0 \qquad \left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)G^i{}_j$$

To show that the equations governing scalars are not affected by tensors, we need to show that tensor perturbation do not contribute to these two components

- $G^0{}_0$ depends on $R^0{}_0$, R. We have already seen that

$$\left(\delta R^0{}_0\right)_{\substack{\text{tensor}\\ \text{1st order}}} = 0 \quad \text{and} \quad \left(\delta R\right)_{\substack{\text{tensor}\\ \text{1st order}}} = 0$$

- $\left(\hat{k}_i\hat{k}_j - \frac{\delta_{ij}}{3}\right)G^i{}_j$

$$\delta G^i{}_j = \delta^{ik}\left(\frac{3}{2}H\frac{\partial \mathcal{H}_{kj}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{kj}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{kj}\right)$$

$$\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)\delta G^i{}_j = \left(\delta_{i3}\delta_{j3} - \frac{1}{3}\delta_{ij}\right)\left(\frac{3}{2}H\frac{\partial \mathcal{H}_{ij}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{ij}\right)$$

$\hat{k} \propto \hat{z}$

$$\cancel{\delta_{i3}\delta_{j3}}\ \delta_{i3}\delta_{j3}(\dots) \propto \cancel{\frac{\partial \mathcal{H}_{33}}{\partial t}}, \cancel{\frac{\partial^2 \mathcal{H}_{33}}{\partial t^2}}, \cancel{\mathcal{H}_{33}}, \ \mathcal{H}_{33} = 0 \implies = 0$$

$$\delta_{ij}(\dots) \propto \cancel{\frac{\partial \mathcal{H}}{\partial t}}, \cancel{\frac{\partial^2 \mathcal{H}}{\partial t^2}}, \cancel{\mathcal{H}}, \ \mathcal{H} \equiv \mathcal{H}_{ii} = 0 \implies = 0$$

So scalar equations are unchanged by the presence of tensor modes!

One could make the same argument for vector mods
$\downarrow$
not considered often!
(need exotic models e.g. topological defects)
Just scalars and tensors

Gauge transformations

Back to scalar perturbations. So far we have worked in a specific gauge/coordinate system

$$ds^2 = -(1+2\Psi)dt^2 + a^2\delta_{ij}(1+2\Phi)\,dx^i dx^j$$

↳ conformal Newtonian gauge

Other well-studied gauges:

- Spatially flat slicing (for inflation) g_{ij} unperturbed
- synchronous ~~pert~~ gauge (perturbations better behaved → numerically) g_{00} unperturbed

But observations are gauge-invariant, and it's useful to know how to move between gauges

General scalar perturbation to the metric

$$g_{00} = -(1+2A)$$

$$g_{0i} = -a\frac{\partial B}{\partial x_i} \qquad \text{(derivative of a scalar function)}$$

$$g_{ij} = a^2\left(\delta_{ij}[1+2\Psi] - 2\frac{\partial^2 E}{\partial x_i \partial x_j}\right)$$

4 functions characterize scalar perturbations: A, B, Ψ, E

Conformal Newtonian gauge: $A=\Psi$, $B=0$, $\Psi=\Phi$, $E=0$

Transform between gauges imposing that invariant distance does not depend on the coordinates that we use

$$x \longrightarrow \tilde{x}$$

$$\tilde{g}_{\alpha\beta}(\tilde{x})\,d\tilde{x}^\alpha d\tilde{x}^\beta = g_{\mu\nu}(x)\,dx^\alpha dx^\beta$$

$$d\tilde{s}^2 = ds^2$$

$$\tilde{g}_{\alpha\beta}(\tilde{x})\, d\tilde{x}^{\alpha}\, d\tilde{x}^{\beta} = \tilde{g}_{\alpha\beta}(\tilde{x}) \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\beta}}{\partial x^{\nu}}\, dx^{\mu} dx^{\nu} = g_{\mu\nu}(x)\, dx^{\mu} dx^{\nu}$$

$$d\tilde{x}^{\alpha} = \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} dx^{\mu} \qquad d\tilde{x}^{\beta} = \frac{\partial \tilde{x}^{\beta}}{\partial x^{\nu}} dx^{\nu}$$

$$\Rightarrow \boxed{\tilde{g}_{\alpha\beta}(\tilde{x}) \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\beta}}{\partial x^{\nu}} = g_{\mu\nu}(x)}$$

How metric changes under a coordinate transformation

Most general coordinate transformation

$$t \longrightarrow \tilde{t} = t + \xi^{0}(t, \bar{x})$$

$$x^{i} \longrightarrow \tilde{x}^{i} = x^{i} + \delta^{iJ} \frac{\partial \xi}{\partial x_J}(t, \bar{x})$$

ξ^0, ξ small perturbations (same order as Ψ, Φ etc.)

Let's work out one component explicitly : 00 $(\mu=0, \nu=0)$

$$\tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^{\alpha}}{\partial x^{0}} \frac{\partial \tilde{x}^{\beta}}{\partial x^{0}} = \tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^{\alpha}}{\partial t} \frac{\partial \tilde{x}^{\beta}}{\partial t} = g_{00}(x) = -[1+2A]$$

$\tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^{\alpha}}{\partial t} \frac{\partial \tilde{x}^{\beta}}{\partial t}$: only $\alpha=\beta=0$ contributes

Example: $\alpha = 0$ $\beta = i$ $\underbrace{\tilde{g}_{0i}}_{\propto \frac{\partial B}{\partial x_i}} \frac{\partial \tilde{x}^{0}}{\partial t} \underbrace{\frac{\partial \tilde{x}^{i}}{\partial t}}_{\propto \frac{\partial \xi}{\partial t \partial x_J}} \propto \frac{\partial B}{\partial x_i} \frac{\partial \xi}{\partial t \partial x_J}$ 2nd order

similarly $\alpha = i$ $\beta = J$

$$\tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^{\alpha}}{\partial t} \frac{\partial \tilde{x}^{\beta}}{\partial t} \underset{\text{1st order}}{\simeq} \tilde{g}_{00} \frac{\partial \tilde{x}^{0}}{\partial t} \frac{\partial \tilde{x}^{0}}{\partial t} = \tilde{g}_{00} \left(\frac{\partial \tilde{t}}{\partial t}\right)^2 = -(1+2\tilde{A})\left(\frac{\partial \tilde{t}}{\partial t}\right)^2 =$$

$$\tilde{t} = t + \xi^{0}$$

$$\frac{\partial \tilde{t}}{\partial t} = 1 + \frac{\partial \xi^{0}}{\partial t}$$

$$= -(1+2\tilde{A})\left(1+\frac{\partial\xi^0}{\partial t}\right)^2 \underset{\uparrow\ \xi^0 \ll 1}{\simeq} -(1+2\tilde{A})\left(1+2\frac{\partial\xi^0}{\partial t}\right) \underset{\uparrow\ \tilde{A}\frac{\partial\xi^0}{\partial A}\ \text{2nd order}}{\simeq} -1-2\tilde{A}-2\frac{\partial\xi^0}{\partial t}$$

$$= g_{00} = -1-2A$$

$$-\cancel{1}-2\tilde{A}-2\frac{\partial\xi^0}{\partial t} = -\cancel{1}-2A \implies -2\tilde{A}-2\frac{\partial\xi^0}{\partial t} = -2A \implies \tilde{A} = A \mp \frac{\partial\xi^0}{\partial t}$$

so under coordinate transformation

$$t \to \tilde{t} = t + \xi^0(t,\vec{x})$$
$$x^i \to \tilde{x}^i \mp x^i + \delta^{ij}\frac{\partial\xi}{\partial x_j}(t,\vec{x})$$

A transforms as

$$A \longrightarrow \tilde{A} = A - \frac{\partial\xi^0}{\partial t} \underset{\uparrow\ d\eta = \frac{dt}{a}\quad \frac{d}{d\eta} = a\frac{d}{dt}\quad \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta}}{=} A - \frac{\dot{\xi}^0}{a}$$

Similarly find how other components transform

$$\tilde{\Psi} = \Psi - H\xi^0$$
$$\tilde{B} = B - \frac{\xi^0}{a} + \dot{\xi}$$
$$\tilde{E} = E + \xi$$

Note: these transformations describe how metric tensor changes the corresponding scalars (e.g. A) don't change in a proper sense, the apparent change is just a redefinition of these scalars!

Although there are 4 functions characterizing scalar perturbations, we can use Gauge invariance to eliminate 2 of them

Example: start with metric where $E \neq 0$

$\longrightarrow$ ~~E~~ Choose $\xi = -E \longrightarrow \tilde{E} = E - \xi = 0$

So really only $4-2=2$ functions are needed to describe scalar perturbations! (e.g. Φ, Ψ)

In fact can construct <u>gauge invariant variables</u> which do not change under gauge transformations. Bardeen variables

$$\Phi_A \equiv A + \frac{1}{a}\frac{\partial}{\partial\eta}\left[a(\dot{E}-B)\right] \longrightarrow \Psi$$

$$\Phi_H \equiv -\psi + aH(B-\dot{E}) \longrightarrow -\Phi$$

conformal Newtonian gauge
$E=B=0$

Useful if we want to calculate perturbations in a gauge where equations become very simple

$T_{\mu\nu}$ also changes under gauge transformations

$$\tilde{T}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}T_{\alpha\beta}(x) \longrightarrow$$ 2 combinations of components of $T_{\mu\nu}$ are gauge-invariant

Fourier space

$$v \equiv ikB + \frac{\hat{k}^i T^0{}_i}{(\rho+P)a}$$

$$\varepsilon_m \equiv -1 - \frac{T^0{}_0}{\rho} + \frac{3H}{k^2\rho}k^i T^0{}_i$$

Conformal Newtonian gauge

$$v = \begin{cases} v_b & b \\ v & DM \\ -3i\Theta_1 & \gamma \end{cases}$$

$3i\Theta_1 \sim$ "v_γ"

$$\varepsilon = \begin{cases} \delta_b + \frac{3aHv_b}{k} & b \\ \delta + \frac{3aHv}{k} & DM \\ 4\Theta_0 - 12i\Theta_1\frac{aH}{k} \end{cases}$$

goes to $\delta, 4\Theta_0$ (standard overdensity) on small scales, $k \to \infty$

<u>Summary of Einstein equations</u>

$$\begin{cases} k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2\left(\rho_{dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0\right) & [\delta G^0{}_0] \\ k^2(\Phi+\Psi) = -32\pi G a^2\left(\rho_\gamma\Theta_2 + \rho_\nu N_2\right) & \left[(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j{}_i)G^i{}_j\right] \end{cases}$$

Scalar perturbations

$$\longrightarrow k^2\Phi = 4\pi G a^2\left[\rho_{dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0 + \frac{3aH}{k}\left(i\rho_{dm}\delta + i\rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0\right)\right]$$

useful combination!

$$\begin{cases} \ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + k^2 h_+ = 0 \\ \ddot{h}_\times + 2\frac{\dot{a}}{a}\dot{h}_\times + k^2 h_\times = 0 \end{cases}$$

tensor perturbations

INITIAL CONDITIONS AND INFLATION

Full Einstein-Boltzmann system for perturbations to the particle distributions and the metric (recap)

$$\left.\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_b\right]\right\}\ \gamma$$

$$\left.\begin{aligned} &\dot{\delta}_{dm} + ikv_{dm} = -3\dot{\Phi} \\ &\dot{v} + \frac{\dot{a}}{a}v = -ik\Psi \end{aligned}\right\}\ \text{DM}$$

$$\left.\begin{aligned} &\cancel{\dot{\delta}_b + \frac{\dot{a}}{a}v_b}\quad \dot{\delta}_b + ikv_b = -3\dot{\Phi} \\ &\dot{v}_b + \frac{\dot{a}}{a}v_b = -ik\Psi + \frac{\dot{\tau}}{R}\left[v_b + 3i\Theta_1\right] \end{aligned}\right\}\ b$$

$$\left.\dot{N} + ik\mu N = -\dot{\Phi} - ik\mu\Psi\right\}\ \nu$$

$$\left.\begin{aligned} &k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2\left(\rho_{dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0\right) \\ &k^2(\Phi + \Psi) = -32\pi G a^2\left(\rho_\gamma\Theta_2 + \rho_\nu N_2\right) \end{aligned}\right\}\ \text{scalar perturbations } \Phi, \Psi$$

$$\left.\begin{aligned} &\ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + k^2 h_+ = 0 \\ &\ddot{h}_\times + 2\frac{\dot{a}}{a}\dot{h}_\times + k^2 h_\times = 0 \end{aligned}\right\}\ \text{tensor perturbations } h_+, h_\times$$

Neglected polarization, assuming massless neutrinos, to 1st order

$4\Theta_0 \sim \delta_\gamma \qquad 4N_0 \sim \delta_\nu \qquad -3i\Theta_1 \sim v_\gamma \qquad -3iN_1 \sim v_\nu$

But what are the initial conditions for $\Theta, \delta, v, N, \Phi, \Psi, h$?

Inflation can explain why the CMB sky is so uniform and provide the initial conditions for the perturbations

Caution: we don't know for sure whether inflation really took place.

The Einstein-Boltzmann equations at early times

It turns out we only really need initial conditions for Φ, then we can relate all other variables to Φ

Consider Boltzmann equations for Θ, δ_{dm}, δ_b, v_{dm}, v_b, $\mathcal{N}$ at very early times : $k\eta \ll 1$ for any k mode we care about

Look at photon equation

$$\underset{①}{\dot{\Theta}} + \underset{②}{ik\mu\Theta} = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_b\right]$$

$① \sim \frac{\Theta}{\eta}$ $\longrightarrow$ $\frac{①}{②} \sim \frac{\Theta}{\eta k \Theta} \sim \frac{1}{k\eta} \gg 1$ by assumption!

$② \sim k\Theta$

Similarly can argue that all terms multiplied by k can be neglected at early times : wavelengths $\sim k^{-1} \gg$ distance over which causal physics operates

↳ observer ~~in than~~ sees only photons from causal horizon with nearly uniform temperature

↳ only monopole survives $\Theta_0 \gg \Theta_1, \Theta_2, \ldots$

$$\dot{\Theta} + \cancel{ik\mu\Theta} = -\dot{\Phi} - \cancel{ik\mu\Psi} - \dot{\tau}\left[\cancel{\Theta_0 - \Theta} + \cancel{\mu v_b}\right]$$

$\Theta_0 - \Theta_0$ $\quad v_b \sim$ 1st order moment ~ 0

$$\dot{\mathcal{N}} + \cancel{ik\mu\mathcal{N}} = -\dot{\Phi} - \cancel{ik\mu\Psi}$$

$$\begin{cases} \dot{\Theta}_0 + \dot{\Phi} = 0 \\ \dot{\mathcal{N}}_0 + \dot{\Phi} = 0 \end{cases}$$

$\Theta \sim \Theta_0$
$\mathcal{N} \sim \mathcal{N}_0$

$$\dot{v}_b + \frac{\dot{a}}{a} v_b = -ik\Psi + \frac{\dot{\tau}}{R}\left[v_b + 3i\Theta_1\right]$$

$\dot{\tau}$ huge $\longrightarrow$ $v_b \simeq -3i\Theta_1$

Set $v \sim 0$ since the vs are basically 1st order moments

$v_b \sim 0$

Even for matter only monopole counts

$\cancel{\dot{\delta}_b + ik v_b}$

$$\dot{\delta}_{dm} + \cancel{ik v_{dm}} = -3\dot{\Phi} \qquad \longrightarrow \qquad \begin{cases} \dot{\delta} = -3\dot{\Phi} \\ \dot{\delta}_b = -3\dot{\Phi} \end{cases}$$
$$\dot{\delta}_b + \cancel{ik v_b} = -3\dot{\Phi}$$

Now consider Einstein equations at early times

$$\cancel{k^2\Phi} + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2 \left(\underbrace{\cancel{\rho_{dm}\delta + \rho_b\delta_b}}_{\text{radiation is dominating}} + 4\rho_\gamma\Theta_0 + 4\rho_\nu N_0\right)$$

$$\longrightarrow 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 16\pi G a^2 \left(\rho_\gamma\Theta_0 + \rho_\nu N_0\right)$$

During radiation domination $a \propto t^{1/2}$

$$\int d\eta = \frac{dt}{a} \propto \int \frac{dt}{t^{1/2}} \sim t^{1/2} \qquad \eta \propto t^{1/2} \propto a \qquad a \propto \eta \qquad a = C\eta \quad \dot{a} = C \quad \frac{\dot{a}}{a} = \frac{1}{\eta}$$

$$\cancel{\frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{d\eta} \propto \frac{1}{a} \propto \frac{1}{\eta}} \qquad \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{d\eta} = \frac{1}{\eta}$$

$$3\frac{1}{\eta}\left(\dot{\Phi} - \frac{\Psi}{\eta}\right) = 16\pi G a^2 \left(\rho_\gamma\Theta_0 + \rho_\nu N_0\right) \longrightarrow \frac{\dot{\Phi}}{\eta} - \frac{\Psi}{\eta^2} = \frac{16\pi G a^2\rho}{3}\left(\frac{\rho_\gamma}{\rho}\Theta_0 + \frac{\rho_\nu}{\rho}N_0\right) \stackrel{*}{=}$$

$$\to \frac{16\pi G a^2 \rho}{3}$$

Friedmann equation $\quad H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho \longrightarrow \left(\frac{\dot{a}}{a}\right)^2 = a^2H^2 = \frac{8\pi G \rho a^2}{3} = \frac{1}{\eta^2}$

$$\longrightarrow \frac{16\pi G a^2 \rho}{3} = \frac{2}{\eta^2}$$

$$\stackrel{*}{=} \frac{2}{\eta^2}\left(\frac{\rho_\gamma}{\rho}\Theta_0 + \frac{\rho_\nu}{\rho}N_0\right) = \left(f_\nu \equiv \frac{\rho_\nu}{\rho_\gamma + \rho_\nu} \qquad f_\gamma = 1 - f_\nu\right.$$

$$= \frac{2}{\eta^2}\left[(1-f_\nu)\Theta_0 + f_\nu N_0\right]$$

$$\longrightarrow \frac{\dot{\Phi}}{\eta} - \frac{\Psi}{\eta^2} = \frac{2}{\eta^2}\left[(1-f_\nu)\Theta_0 + f_\nu N_0\right]$$

$$\longrightarrow \dot{\Phi}\eta - \Psi = 2\left[(1-f_\nu)\Theta_0 + f_\nu N_0\right]$$

But we also know that

$$\dot{\Theta}_0 + \dot{\Phi} = 0$$
$$\dot{N}_0 + \dot{\Phi} = 0$$

$$\dot{\Phi}\eta - \Psi = 2\left([1-f_\nu]\Theta_0 + f_\nu N_0\right)$$

Differentiate both sides

$$\ddot{\Phi}\eta + \dot{\Phi} - \dot{\Psi} = 2\left([1-f_\nu]\dot{\Theta}_0 + f_\nu \dot{N}_0\right)$$

$\Downarrow$ $\dot{\Theta}_0 = -\dot{\Phi}$, $\dot{N}_0 = -\dot{\Phi}$

$$(1-f_\nu)\dot{\Theta}_0 + f_\nu \dot{N}_0 = -\dot{\Phi} + \cancel{f_\nu\dot{\Phi}} - \cancel{f_\nu\dot{\Phi}} = -\dot{\Phi}$$

$$\ddot{\Phi}\eta + \dot{\Phi} - \dot{\Psi} = -2\dot{\Phi}$$

Look at the other Einstein equation

neglect higher-order moments

$$k^2(\Phi + \Psi) = -32\pi G a^2 \left(\cancel{\rho_\gamma \Theta_2} + \cancel{\rho_\nu N_2}\right) \Rightarrow \Psi \simeq -\Phi$$

$$\ddot{\Phi}\eta + \dot{\Phi} + \underbrace{\dot{\Phi}}_{-(-\dot{\Phi})} = -2\dot{\Phi}$$

$$\boxed{\ddot{\Phi}\eta + 4\dot{\Phi} = 0}$$

Solve by $\Phi = \eta^p \Rightarrow \ddot{\Phi} = p(p-1)\eta^{p-2}$ $\quad \dot{\Phi} = p\eta^{p-1}$

$$\rightarrow p(p-1)\cancel{\eta^{p-1}} + 4p\cancel{\eta^{p-1}} = 0 \quad \rightarrow p(p-1) + 4p = 0$$

$$p^2 + 3p = p(p+3) = 0 \Rightarrow p = 0 \quad p = -3$$

$p=0$: growing mode which we care about

$p=-3$: decaying mode cosmologically irrelevant!

Somehow we need to excite the growing mode

$$\Phi = \eta^0 = \text{const} \quad \dot{\Phi} = 0 \quad \Psi = -\Phi$$

$$\Phi = 2\left([1-f_\nu]\Theta_0 + f_\nu N_0\right)$$

$$\Leftrightarrow \Theta_0 , N_0 = \text{const}$$

In most models of structure formation not only are Θ_0 and $\mathcal{N}_0$ constant but also equal

$$\Theta_0(k,\eta_i) = \mathcal{N}_0(k,\eta_i)$$

$$\Rightarrow \Phi = 2\left[(1-f_\nu)\Theta_0 + f_\nu \mathcal{N}_0\right] \overset{\Theta_0=\mathcal{N}_0}{=} 2\Theta_0$$

$$\hookrightarrow \boxed{\Phi(k,\eta_i) = 2\Theta_0(k,\eta_i)}$$

Note:
- explicit k-dependence
- initial conditions set at very early η_i

Back to

$$\dot\delta = -3\dot\Phi \qquad \dot\Theta_0 + \dot\Phi = 0 \Longrightarrow \delta = 3\Theta_0 + \text{const}$$
$$\dot\delta_b = -3\dot\Phi \qquad\qquad\qquad\qquad \delta_b = 3\Theta_0 + \text{const}$$

Two types of primordial perturbations

const $=0 \rightarrow$ ADIABATIC

const $\neq 0 \rightarrow$ ISOCURVATURE $\longrightarrow$ severely contrained!

Adiabatic perturbations: constant number density ratio between all species <u>everywhere</u>

$$\frac{n_{dm}}{n_\gamma} = \underbrace{\frac{n_{dm}^{(0)}}{n_\gamma^{(0)}}}_{\text{constant in space and time}} \underbrace{\left[\frac{1+\delta}{1+3\Theta_0}\right]}_{\simeq 1+\delta-3\Theta_0 = 1 \Longrightarrow \delta = 3\Theta_0}$$

to be independent of space and time

So $\delta = 3\Theta_0$ and similarly $\delta_b = 3\Theta_0$

Initial conditions for velocities (exercise)

$$\Theta_1 = \mathcal{N}_1 = \frac{i v_b}{3} = \frac{i v}{3} = -\frac{k\Phi}{6aH}$$

Summarizing IC: $\Phi = -\Psi = 2\Theta_0 \quad \delta = \delta_b = 3\Theta_0 \quad \Theta_1 = \mathcal{N}_1 = \frac{iv_b}{3} = \frac{iv}{3} = -\frac{k\Phi}{6aH}$