$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j{}_i\right)G^i{}_j = \left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j{}_i\right)\left[A\delta^i{}_j + \frac{k^ik_j(\Phi+\Psi)}{a^2}\right]$$

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j{}_i\right)A\delta^i{}_j = A\underbrace{\hat{k}_i\hat{k}^j\delta^i{}_j}_{|\hat{k}|^2=1} - \frac{1}{3}A\underbrace{\delta^j{}_i\delta^i{}_j}_{3} = A - \frac{3}{3}A = A - A = 0!$$

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j{}_i\right)\frac{k^ik_j(\Phi+\Psi)}{a^2} = \underbrace{\hat{k}_i\hat{k}^jk^ik_j}_{k^2}\frac{(\Phi+\Psi)}{a^2} - \frac{1}{3}\underbrace{\delta^j{}_ik^ik_j}_{k^2}\frac{(\Phi+\Psi)}{a^2} =$$

$$= \frac{k^2(\Phi+\Psi)}{a^2} - \frac{1}{3}k^2\frac{(\Phi+\Psi)}{a^2} = \frac{2}{3}k^2\frac{(\Phi+\Psi)}{a^2}$$

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j{}_i\right)G^i{}_j = \frac{2k^2(\Phi+\Psi)}{3a^2} = 8\pi G\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j{}_i\right)T^i{}_j$$

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j{}_i\right)T^i{}_j = \sum_i\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j{}_i\right)P_i =$$

$P_i = g_i\int\frac{d^3p}{(2\pi)^3}f_i\frac{p^2}{3E(p)}$

$$= \sum_i g_i\int\frac{d^3p}{(2\pi)^3}\frac{p^2\mu^2 - \frac{1}{3}p^2}{E_i(p)}f_i(\vec{p})$$

$\hat{k}_i\hat{k}^jp^2 = p^2\mu^2$

$\delta^j{}_ip^2 = p^2$

$i \to \gamma, \nu$ since only γ, ν have non-zero quadrupole

$p^2\mu^2 - \frac{1}{3}p^2 = \frac{2}{3}P_2(\mu)p^2$ since $P_2(\mu) = \frac{1}{2}(3\mu^2-1)$ ~~Legendre poly~~

picks out quadrupole!

0th order distribution no quadrupole

Look at photons

$$g_\gamma\int\frac{d^3p}{(2\pi)^3}\frac{2}{3}P_2(\mu)\frac{p^2}{E}f(p) \simeq g_\gamma\int\frac{d^3p}{(2\pi)^3}\frac{2}{3}P_2(\mu)p\left[\cancel{f^{(0)}} - p\frac{\partial f^{(0)}}{\partial p}\Theta\right]$$

$p = E$

$$= -2\underbrace{\int\frac{dp\,p^2}{2\pi^2}}\,p^2\frac{\partial f^{(0)}}{\partial p}\int_{-1}^{1}\frac{d\mu}{2}\frac{2P_2(\mu)}{3}\Theta(\mu) = \frac{2\Theta_2}{3}2\int\frac{dp\,p^2}{2\pi^2}p^2\frac{\partial f^{(0)}}{\partial p}$$

$g_\gamma = 2$

$$\int\frac{d^3p}{(2\pi)^3} = \int\frac{dp\,p^2}{8\pi^3}\underbrace{\int d\varphi}_{2\pi}\int d\mu = \int\frac{dp\,p^2\,2\pi}{8\pi^3}\int d\mu = \int dp\frac{p^2}{4\pi^2}\int d\mu = \int dp\frac{p^2}{2\pi^2}\int\frac{d\mu}{2}$$

$$\Theta_2 \equiv -\int_{-1}^{1}\frac{d\mu}{2}P_2(\mu)\Theta(\mu)$$

$$\frac{2\Theta_2}{3}\, 2\int \frac{dp\, p^2}{2\pi^2}\, p^2 \frac{\partial f^{(0)}}{\partial p} \underset{\text{by parts}}{=} -\frac{2\Theta_2}{3}\, 4\times \underbrace{2\int \frac{dp\, p^3}{2\pi^2} f^{(0)}}_{} = -\frac{8\Theta_2}{3}\rho_\gamma$$

$$= \underset{g_\gamma}{2}\int \frac{d^3p}{(2\pi)^3}\, p f^{(0)} = 2\int \frac{d^3p}{(2\pi)^3}\, E f^{(0)} = \rho_\gamma$$

anisotropic stress: only ___relativistic___ particles contribute to the anisotropic stress of $T_{\mu\nu}$

Putting everything together...

~~$(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i^{\,j})\, T^i_{\ j} = ($~~

$$\frac{2}{\cancel{3}a^2} k^2(\Phi+\Psi) = (\hat{k}_i \hat{k}^j - \tfrac{1}{3}\delta_i^{\,j})\, G^i_{\ j} = 8\pi G(\hat{k}_i \hat{k}^j - \tfrac{1}{3}\delta_i^{\,j})\, T^i_{\ j} =$$

$$= -8\pi G \times \frac{\cancel{8}\,4}{\cancel{3}}(\rho_\gamma \Theta_2 + \rho_\nu \mathcal{N}_2)$$

$$\Rightarrow \boxed{k^2(\Phi+\Psi) = -32\pi G a^2[\rho_\gamma \Theta_2 + \rho_\nu \mathcal{N}_2]}$$

$\to$ $\Phi = -\Psi$ __unless__ relativistic species have appreciable quadrupole moments (γ, ν)

In practice because of tight coupling of photons to baryons (mostly Θ_0, Θ_1) the sum is in large part due to neutrinos

Recap: 2 components of Einstein equations for __scalar__ perturbations

$$\boxed{\begin{aligned} &k^2\Phi + 3\frac{\dot a}{a}\left(\dot\Phi - \Psi\frac{\dot a}{a}\right) = 4\pi G a^2[\rho_{dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0] \\ &k^2(\Phi+\Psi) = -32\pi G a^2[\rho_\gamma\Theta_2 + \rho_\nu\mathcal{N}_2] \end{aligned}}$$

Tensor perturbations

So far we looked at scalar perturbations

$\Phi(\bar{x},t)$, $\Psi(\bar{x},t)$ are unchanged under $\bar{x} \to \bar{x}'$

Scalar perturbations $\xleftrightarrow[\text{are sourced by}]{\text{source/}}$ density fluctuations

We mostly care about these

But many theories of structure formation (e.g. inflation) also predict tensor perturbations! Can study using the same tools we used for scalar perturbations

But why can we consider scalar and tensor perturbations separately?

Because they decouple! We will see this explicitly

↳ evolve independently

↳ Manifestation of the decomposition theorem (more later)

Tensor perturbations to FLRW metric : h_+, $h_\times$

$$g_{00} = -1 \qquad g_{0i} = 0$$

$$g_{ij} = a^2 \begin{pmatrix} 1+h_+ & h_\times & 0 \\ h_\times & 1-h_+ & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

perturbation axis in xy plane

$\bar{k}$ wavevector along z-axis

h_+, $h_\times$ components of a divergenceless, traceless, symmetric tensor H_{ij}
① ② ③

$$g_{ij} = g_{ij}^{(0)} + H_{ij} \qquad H_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

① $k^i H_{ij} = k^j H_{ij} = 0$ self-evident since no components in $\hat{k} = \hat{z}$ direction

② $\delta^{ij} H_{ij} = 0$ $\qquad \delta^{ij} H_{ij} = h_+ - h_+ + 0 = 0$

③ $H_{ij} = H_{ji}$ self-evident since apart from $h_\times$ H_{ij} is diagonal

Now follow same steps as before for scalar perturbations:

1) $\Gamma^{\mu}_{\alpha\beta}$ from perturbed metric

2) $R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$

3) $R = g^{\mu\nu} R_{\mu\nu}$

Christoffel symbols

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\beta\nu} - \partial_\nu g_{\alpha\beta}\right]$$

$g_{\mu\nu}$ has constant g_{00}, and $g_{0i} = 0$ $\rightarrow$ only terms non-zero involve derivatives of g_{ij}

$$\cancel{\mu = 0 \rightarrow \nu = 0 \quad \Gamma \sim \partial_\beta g_{\alpha 0} + \partial_\alpha g_{\beta 0} - \partial_0 g_{\alpha\beta}}$$

$$\Gamma^{0}_{00} \sim \cancel{\partial_0 g_{00}} + \cancel{\partial_0 g_{00}} - \cancel{\partial_0 g_{00}}$$

$$\Gamma^{0}_{i0} \sim \cancel{\partial_0 g_{i0}} + \cancel{\partial_i g_{00}} - \cancel{\partial_0 g_{i0}}$$

$$\Gamma^{0}_{00} = \Gamma^{0}_{i0} = 0$$

$$\Gamma^{0}_{ij} = \frac{g^{0\nu}}{2}\left[\partial_j g_{i\nu} + \partial_i g_{j\nu} - \partial_\nu g_{ij}\right] \overset{\nu=0}{=} \frac{g^{00}}{2}\left[\cancel{\partial_j g_{i0}} + \cancel{\partial_i g_{j0}} - \partial_0 g_{ij}\right] =$$

$g^{00} = (g_{00})^{-1} = -1$

$$= \frac{1}{2}\partial_0 g_{ij} \qquad g_{ij} = a^2(\delta_{ij} + \mathcal{H}_{ij}) \qquad \mathcal{H}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\cancel{\partial_0 g_{ij} = \frac{\partial g_{ij}}{\partial t} = \frac{\partial}{\partial t}(a^2 \mathcal{H}_{ij}) = 2\dot{a}a\mathcal{H}_{ij} + a^2\partial_0 \mathcal{H}_{ij} = 2\frac{\dot{a}}{a}a^2\mathcal{H}_{ij}}$$

$$\partial_0 g_{ij} = \frac{\partial g_{ij}}{\partial t} = \frac{\partial}{\partial t}\left[a^2(\delta_{ij} + \mathcal{H}_{ij})\right] = 2\dot{a}a(\delta_{ij} + \mathcal{H}_{ij}) + a^2\frac{\partial \mathcal{H}_{ij}}{\partial t} =$$

$$= 2\underbrace{\frac{\dot{a}}{a}}_{H}\underbrace{a^2(\delta_{ij} + \mathcal{H}_{ij})}_{g_{ij}} + a^2\frac{\partial \mathcal{H}_{ij}}{\partial t} = 2Hg_{ij} + \frac{\partial \mathcal{H}_{ij}}{\partial t}$$

$$\Rightarrow \Gamma^0_{ij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial t} = H g_{ij} + \frac{a^2}{2}\frac{\partial \mathcal{H}_{ij}}{\partial t}$$

$$\Gamma^\mu_{00} = \frac{g^{\mu\nu}}{2}\left[\cancel{\partial_0 g_{0\nu}} + \cancel{\partial_0 g_{0\nu}} - \cancel{\partial_\nu g_{00}}\right] = 0 \quad \forall \mu$$

$$\Gamma^i_{0j} = \frac{g^{ik}}{2}\left[\cancel{\partial_j g_{0k}}^{\,g_{0k}=0} + \partial_0 g_{jk} - \cancel{\partial_k g_{0j}}^{\,g_{0j}=0}\right] = \cancel{\tfrac{1}{2} g^{ik} \partial_k g_{0j}}\ \frac{1}{2} g^{ik}\partial_0 g_{jk} =$$

$$= \frac{1}{2} g^{ik}\frac{\partial}{\partial t} g_{jk}$$

$$\frac{\partial}{\partial t} g_{jk} = \frac{\partial}{\partial t}\left[a^2(\delta_{jk} + \mathcal{H}_{jk})\right] = 2H g_{jk} + a^2 \frac{\partial}{\partial t}\mathcal{H}_{jk}$$

same steps as before

$$\Gamma^i_{0j} = \frac{1}{2} g^{ik}\frac{\partial}{\partial t} g_{jk} = \frac{1}{2} g^{ik}\left(2H g_{jk} + a^2\frac{\partial}{\partial t}\mathcal{H}_{jk}\right) = H\delta_{ij} + \frac{1}{2}\frac{\partial \mathcal{H}_{ij}}{\partial t}$$

$\mathcal{H}_{ij} = \mathcal{H}_{ji}$

Γ^i_{jk} = (exercise) $\quad \frac{i}{2}\left[k_k \mathcal{H}_{ij} + k_j \mathcal{H}_{ik} - k_i \mathcal{H}_{jk}\right]$ (obviously in Fourier space)

In summary

$$\Gamma^0_{00} = 0 \qquad \Gamma^0_{i0} = 0 \qquad \Gamma^i_{00} = 0 \qquad \Gamma^0_{ij} = H g_{ij} + \frac{a^2}{2}\frac{\partial \mathcal{H}_{ij}}{\partial t}$$

$$\Gamma^i_{0j} = H\delta_{ij} + \frac{1}{2}\frac{\partial \mathcal{H}_{ij}}{\partial t} \qquad \Gamma^i_{jk} = \frac{i}{2}\left[k_k \mathcal{H}_{ij} + k_j \mathcal{H}_{ik} - k_i \mathcal{H}_{jk}\right]$$

Ricci tensor

$$R_{00} = \cancel{\partial_\alpha \Gamma^\alpha_{00}}^{\,\Gamma^\alpha_{00}=0} - \partial_0 \Gamma^\alpha_{0\alpha} + \cancel{\Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00}}^{\,\Gamma^\beta_{00}=0} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha}$$

$\alpha, \beta = i$

$$\Rightarrow R_{00} = -\partial_0 \Gamma^i_{0i} - \Gamma^i_{j0}\Gamma^j_{0i} = -\frac{\partial}{\partial t}\Gamma^i_{0i} - \Gamma^i_{j0}\Gamma^j_{0i}$$

$$\frac{\partial}{\partial t}\Gamma^i_{0i} = \frac{\partial}{\partial t}\left(H\underbrace{\delta_{ii}}_{3} + \cancel{\frac{1}{2}\frac{\partial}{\partial t}\mathcal{H}_{ii}}^{\,\mathcal{H}_{ii}=0}\right) = 3\frac{dH}{dt}$$

$$\Rightarrow R_{00} = -3\frac{dH}{dt} - \left(H\delta_{ij} + \frac{1}{2}\frac{\partial}{\partial t}\mathcal{H}_{ij}\right)\left(H\delta_{ji} + \frac{1}{2}\frac{\partial}{\partial t}\mathcal{H}_{ji}\right) =$$

$$= -3\frac{dH}{dt} - H^2 \underbrace{\delta_{ii}}_{3} - \frac{H}{2}\delta_{ij}\cancel{\frac{\partial}{\partial t}\mathcal{H}_{ij}} - \frac{H}{2}\delta_{ij}\cancel{\frac{\partial}{\partial t}\mathcal{H}_{ij}} - \frac{1}{4}\cancel{\left(\frac{\partial}{\partial t}\mathcal{H}_{ij}\right)^2}$$

$\delta_{ij}\mathcal{H}_{ij} = 0$ $\delta_{ij}\mathcal{H}_{ij} = 0$ 2nd order

$$\simeq -3\left(\frac{dH}{dt} + H^2\right) = -3\frac{d^2a/dt^2}{a}$$ already found earlier!

$$\frac{dH}{dt} = \frac{d}{dt}\left(\frac{1}{a}\frac{da}{dt}\right) = \frac{1}{a}\frac{d^2a}{dt^2} - \frac{1}{a^2}\frac{da}{dt}\frac{da}{dt} = \frac{1}{a}\frac{d^2a}{dt^2} - \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{1}{a}\frac{d^2a}{dt^2} - H^2$$

$$\hookrightarrow \frac{d^2a}{dt^2} = a\left(\frac{dH}{dt} + H^2\right)$$

$$\hookrightarrow \frac{dH}{dt} + H^2 = \frac{1}{a}\frac{d^2a}{dt^2}$$

$(R_{00})_{1st} = 0$ for tensor perturbations!

Recall for scalar perturbations $(R_{00})_{1st} = -\frac{k^2}{a^2}\Psi - 3\frac{\partial^2\Phi}{\partial t^2} + 3H\left(\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Phi}{\partial t}\right)$

First manifestation of the decomposition theorem

$\hookrightarrow$ time-time component of Einstein's equations does not contain tensor perturbations

$\hookrightarrow$ density perturbations (RHS of $(\delta G^0{}_0)_{1st}$) do not induce tensor perturbations.

Tensor perturbations evolve on their own

$$R_{ij} = \partial_\alpha \Gamma^\alpha{}_{ij} - \partial_j \Gamma^\alpha{}_{i\alpha} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{ij} - \Gamma^\alpha{}_{\beta j}\Gamma^\beta{}_{i\alpha}$$

$\Gamma^0{}_{i0} = 0$

$$\partial_\alpha \Gamma^\alpha{}_{ij} - \partial_j \Gamma^\alpha{}_{i\alpha} = \underset{①}{\partial_0 \Gamma^0{}_{ij}} + \underset{②}{\partial_k \Gamma^k{}_{ij}} - \cancel{\partial_j \Gamma^0{}_{i0}} - \underset{③}{\partial_j \Gamma^k{}_{ik}}$$

① $\partial_0 \Gamma^0{}_{ij} = \frac{\partial}{\partial t}\left(Hg_{ij} + \frac{a^2}{2}\frac{\partial \mathcal{H}_{ij}}{\partial t}\right) = \frac{1}{2}\frac{\partial^2}{\partial t^2} g_{ij}$ calculate later

③ $\partial_j \Gamma^k{}_{ik} = 0$ since $\Gamma^k{}_{ik} = \frac{i}{2}\left[\cancel{k_k \mathcal{H}_{ki}} + \cancel{k_i \mathcal{H}_{kk}} - \cancel{k_k \mathcal{H}_{ik}}\right] = 0$

② $\partial_k \Gamma^k_{ij} = \cancel{\frac{i}{2}} ik_k \left[\frac{i}{2}\left(k_j \mathcal{H}_{ki} + k_i \mathcal{H}_{kj} - k_k \mathcal{H}_{ij} \right) \right] =$

$\partial_k \to ik_k$

$$= \left(-\frac{k_k k_j \mathcal{H}_{ki}}{2} - \frac{k_k k_i \mathcal{H}_{kj}}{2} + \frac{k^2}{2} \mathcal{H}_{ij} \right) = \frac{1}{2}\Big(\underbrace{-k_i k_k \mathcal{H}_{jk}}_{i=k=3} \overset{a)}{} - \underbrace{k_j k_k \mathcal{H}_{ik}}_{j=k=3} \overset{b)}{} + k^2 \mathcal{H}_{ij} \Big)$$

since $\vec{k}$ along z-axis

a) $k_i k_k \mathcal{H}_{jk} = k_3 k_3 \mathcal{H}_{j3} = 0$

b) $k_j k_k \mathcal{H}_{ik} = k_3 k_3 \mathcal{H}_{i3} = 0$ since $\mathcal{H} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\to 0!$

$\mathcal{H}_{j3} = \mathcal{H}_{3j} = 0$

$$\Rightarrow \partial_k \Gamma^k_{ij} = \frac{k^2}{2} \mathcal{H}_{ij}$$

$$\Rightarrow \partial_\alpha \Gamma^\alpha_{ij} - \partial_j \Gamma^\alpha_{i\alpha} = \frac{\partial^2 g_{ij}}{\partial t^2} + \frac{k^2}{2} \mathcal{H}_{ij}$$

$\to \Gamma^\alpha_{\alpha\beta} \Gamma^\beta_{ij} \neq 0$ only if $\alpha = k$

2nd order ; $\mathcal{H}_{kk} = 0$

$$\Gamma^k_{k\beta} \Gamma^\beta_{ij} = \Gamma^k_{k0} \Gamma^0_{ij} + \cancel{\Gamma^k_{kl} \Gamma^l_{ij}} \simeq \Gamma^k_{k0} \Gamma^0_{ij} = \left(H \delta_{kk} + \cancel{\frac{1}{2} \frac{\partial \mathcal{H}_{kk}}{\partial t}} \right) \underbrace{\left(H g_{ij} + \frac{a^2}{2} \frac{\partial \mathcal{H}_{ij}}{\partial t} \right)}_{\frac{1}{2}\frac{\partial g_{ij}}{\partial t}}$$

$$= 3H \times \frac{1}{2} \frac{\partial g_{ij}}{\partial t} = \frac{3}{2} H \frac{\partial g_{ij}}{\partial t}$$

$\Gamma^\alpha_{\beta j} \Gamma^\beta_{i\alpha} =$ (exercise) $\quad 2H^2 g_{ij} + 2a^2 H \frac{\partial \mathcal{H}_{ij}}{\partial t}$

Putting everything together...

$$R_{ij} = \partial_\alpha \Gamma^\alpha_{ij} - \partial_j \Gamma^\alpha_{i\alpha} + \Gamma^\alpha_{\alpha\beta} \Gamma^\beta_{ij} - \Gamma^\alpha_{\beta j} \Gamma^\beta_{i\alpha} =$$

$$= \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial t^2} + \frac{k^2}{2} \mathcal{H}_{ij} + \frac{3}{2} H \frac{\partial g_{ij}}{\partial t} - 2H^2 g_{ij} + 2a^2 H \frac{\partial \mathcal{H}_{ij}}{\partial t}$$

To make progress we need to expand $\frac{\partial g_{ij}}{\partial t}$ and $\frac{\partial^2 g_{ij}}{\partial t^2}$

~~We already saw $\frac{\partial g_{ij}}{\partial t} = 2H g_{ij} + a^2 \mathcal{H}_{ij,0}$~~

We already saw

$$\frac{\partial g_{ij}}{\partial t} = \frac{\partial}{\partial t}\left[a^2(\delta_{ij} + h_{ij})\right] = 2H g_{ij} + a^2 \frac{\partial h_{ij}}{\partial t}$$

$$\frac{\partial^2 g_{ij}}{\partial t^2} = \frac{\partial}{\partial t}\left[2H g_{ij} + a^2 \frac{\partial h_{ij}}{\partial t}\right] = 2\frac{dH}{dt} g_{ij} + 2H\frac{\partial g_{ij}}{\partial t} + 2a\frac{da}{dt}\frac{\partial h_{ij}}{\partial t} + a^2 \frac{\partial^2 h_{ij}}{\partial t^2}$$

$$= 2\left(\frac{1}{a}\frac{d^2 a}{dt^2} - H^2\right) g_{ij} + 2H\left(2H g_{ij} + a^2 \frac{\partial h_{ij}}{\partial t}\right) + 2Ha^2 \frac{\partial h_{ij}}{\partial t} + a^2 \frac{\partial^2 h_{ij}}{\partial t^2} =$$

$\frac{dH}{dt} = \frac{1}{a}\frac{d^2 a}{dt^2} - H^2$

$a\frac{da}{dt} = a^2 H$

$$= 2g_{ij}\left(\frac{1}{a}\frac{d^2 a}{dt^2} - H^2\right) + 4H^2 g_{ij} + 2Ha^2 \frac{\partial h_{ij}}{\partial t} + 2Ha^2 \frac{\partial h_{ij}}{\partial t} + a^2 \frac{\partial^2 h_{ij}}{\partial t^2} =$$

$+$ $\quad$ $+$

$$= 2g_{ij}\left(\frac{d^2a/dt^2}{a} + H^2\right) + 4Ha^2 \frac{\partial h_{ij}}{\partial t} + a^2 \frac{\partial^2 h_{ij}}{\partial t^2}$$

$$\Rightarrow R_{ij} = \frac{1}{2}\frac{\partial^2 g_{ij}}{\partial t^2} + \frac{k^2}{2} h_{ij} + \frac{3}{2} H \frac{\partial g_{ij}}{\partial t} - 2H^2 g_{ij} + 2a^2 H \frac{\partial h_{ij}}{\partial t} =$$

$$= g_{ij}\left(\frac{d^2a/dt^2}{a} + H^2\right) + 2Ha^2 \frac{\partial h_{ij}}{\partial t} + \frac{1}{2}a^2 \frac{\partial^2 h_{ij}}{\partial t^2} + \frac{k^2}{2} h_{ij} + \frac{3}{2} H\left(2H g_{ij} + a^2 \frac{\partial h_{ij}}{\partial t}\right)$$

$$- 2H^2 g_{ij} + 2a^2 H \frac{\partial h_{ij}}{\partial t} =$$

$$= g_{ij}\left(\frac{d^2a/dt^2}{a} + H^2\right) + \cancel{2Ha^2 \frac{\partial h_{ij}}{\partial t}} + \frac{1}{2}a^2 \frac{\partial^2 h_{ij}}{\partial t^2} + \frac{k^2}{2} h_{ij} + 3H^2 g_{ij} + \frac{3}{2} Ha^2 \frac{\partial h_{ij}}{\partial t}$$

$$- 2H^2 g_{ij} - \cancel{2a^2 H \frac{\partial h_{ij}}{\partial t}} =$$

$$= \underbrace{g_{ij}\left(\frac{d^2a/dt^2}{a} + 2H^2\right)}_{\text{0th order part of } R_{ij} \text{ is correct}} + \frac{3}{2}a^2 H \frac{\partial h_{ij}}{\partial t} + \frac{1}{2}a^2 \frac{\partial^2 h_{ij}}{\partial t^2} + \frac{k^2}{2} h_{ij}$$

Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij}$$

R_{ij} given previously, also $R_{00} = -3\frac{d^2a/dt^2}{a}$ (0th order)

$g^{00} = -1$ for tensor perturbations

$g^{00}R_{00} = 3\frac{d^2a/dt^2}{a}$ → purely 0th order

$$g^{ij}R_{ij} = g^{ij}\Big[\underbrace{A}_{\left(\frac{d^2a/dt^2}{a}+2H^2\right)} g_{ij} + \underbrace{\dots\left(\mathcal{H}_{ij}, \frac{\partial \mathcal{H}_{ij}}{\partial t}, \frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2}\right)}_{\text{1st order, set } g^{ij} = \frac{\delta_{ij}}{a^2}}\Big] =$$

$$= \underbrace{3}_{g^{ij}g_{ij}}\left(\frac{d^2a/dt^2}{a} + 2H^2\right) + \underbrace{\propto \cancel{g^{ij}\mathcal{H}_{ij}},\ \cancel{g^{ij}\frac{\partial \mathcal{H}_{ij}}{\partial t}},\ \cancel{g^{ij}\frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2}}}_{\text{trace} \quad \delta^{ij}\mathcal{H}_{ij} = 0}$$

Putting everything together

$$R = 3\frac{d^2a/dt^2}{a} + 3\left(\frac{d^2a/dt^2}{a} + 2H^2\right) = 6\underbrace{\left(\frac{d^2a/dt^2}{a} + H^2\right)}_{\text{standard result}} \quad \text{0th order!}$$

Tensor perturbations do not affect Ricci scalar at first order

↳ again manifestation of decomposition theorem

Einstein equations for tensor perturbations

$\delta G^i{}_j = \delta R^i{}_j$ (since R unperturbed)

$R^i{}_j = g^{ik}R_{kj}$: look only at 1st order terms

$$= \overset{\text{0th order}}{\cancel{\underbrace{g^{ik}g_{ij}}_{\delta^k{}_j}\left(\frac{d^2a/dt^2}{a} + 2H^2\right)}} + \underbrace{g^{ik}}_{\delta^{ik}/a^2}\frac{3}{2}a^2 H\frac{\partial \mathcal{H}_{kj}}{\partial t} + \underbrace{g^{ik}}_{\frac{\delta^{ik}}{a^2}}\frac{a^2}{2}\frac{\partial^2 \mathcal{H}_{kj}}{\partial t^2} + \underbrace{g^{ik}}_{\frac{\delta^{ik}}{a^2}}\frac{k^2}{2}\mathcal{H}_{kj}$$

$$\Rightarrow \left(\delta G^i{}_j\right)_{\substack{\text{tensor}\\ \text{1st order}}} = \delta^{ik}\left(\frac{3}{2} H \frac{\partial \mathcal{H}_{kj}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{kj}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{kj}\right)$$

From these we need two equations for $h_+, h_\times$

Recall $\mathcal{H}_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\qquad \mathcal{H}_{11} = h_+ = -\mathcal{H}_{22}$

Since $\delta G^i{}_j$ involves $\mathcal{H}_{ij}, \frac{\partial \mathcal{H}_{ij}}{\partial t}, \frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2} \Longrightarrow \delta G^1{}_1 = -\delta G^2{}_2$

$$\delta G^1{}_1 = \delta^{1k}\left[\frac{3}{2}H\frac{\partial \mathcal{H}_{k1}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{k1}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{k1}\right] = \frac{3}{2}H\frac{\partial \mathcal{H}_{11}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{11}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{11}$$

$$\delta G^2{}_2 = \delta^{2k}\left[\frac{3}{2}H\frac{\partial \mathcal{H}_{k2}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{k2}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{k2}\right] = \frac{3}{2}H\frac{\partial \mathcal{H}_{22}}{\partial t} + \frac{1}{2}\frac{\partial^2 \mathcal{H}_{22}}{\partial t^2} + \frac{k^2}{2a^2}\mathcal{H}_{22}$$

$$\Downarrow$$

$$\delta G^1{}_1 - \delta G^2{}_2 = \frac{3}{2}H\underbrace{\left(\frac{\partial \mathcal{H}_{11}}{\partial t} - \frac{\partial \mathcal{H}_{22}}{\partial t}\right)}_{2\frac{\partial \mathcal{H}_{11}}{\partial t} = 2\frac{\partial h_+}{\partial t}} + \frac{1}{2}\underbrace{\left(\frac{\partial^2 \mathcal{H}_{11}}{\partial t^2} - \frac{\partial^2 \mathcal{H}_{22}}{\partial t^2}\right)}_{2\frac{\partial^2 \mathcal{H}_{11}}{\partial t^2} = 2\frac{\partial^2 h_+}{\partial t^2}} + \frac{k^2}{2a^2}\underbrace{\left(\mathcal{H}_{11} - \mathcal{H}_{22}\right)}_{2\mathcal{H}_{11} = 2h_+} =$$

$$= 3H\frac{\partial h_+}{\partial t} + \frac{\partial^2 h_+}{\partial t^2} + \frac{k^2}{a^2}h_+$$

Change to conformal time

$$d\eta = \frac{dt}{a} \rightarrow \frac{d}{d\eta} = a\frac{d}{dt} \rightarrow \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta}$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{1}{a}\frac{d}{d\eta}\right) = \frac{1}{a}\frac{d}{d\eta}\left(\frac{1}{a}\frac{d}{d\eta}\right) = \frac{1}{a^2}\frac{d^2}{d\eta^2} - \frac{1}{a}\frac{\mathring{a}}{a^2}\frac{d}{d\eta} = \frac{1}{a^2}\frac{d^2}{d\eta^2} - \frac{\mathring{a}}{a^3}\frac{d}{d\eta}$$

$$\Rightarrow \frac{\partial h_+}{\partial t} = \frac{1}{a}\frac{dh_+}{d\eta} = \frac{\mathring{h}_+}{a} \qquad \frac{\partial^2 h_+}{\partial t^2} = \frac{1}{a^2}\frac{d^2 h_+}{d\eta^2} - \frac{\mathring{a}}{a^3}\frac{dh_+}{d\eta} = \frac{\ddot{h}_+}{a^2} - \frac{\mathring{a}}{a^3}\mathring{h}_+$$

$$\Rightarrow \delta G^1{}_1 - \delta G^2{}_2 = 3H\frac{\partial h_+}{\partial t} + \frac{\partial^2 h_+}{\partial t^2} + \frac{k^2 h_+}{a^2} = 3H\frac{\mathring{h}_+}{a} + \frac{\ddot{h}_+}{a^2} - \frac{\mathring{a}}{a^3}\mathring{h}_+ + \frac{k^2 h_+}{a^2} =$$

$$H = \frac{1}{a}\frac{da}{dt} = \frac{1}{a^2}\frac{da}{d\eta} = \frac{\mathring{a}}{a^2}$$

$$= 3\frac{\dot a}{a^2}\frac{\dot h_+}{a} + \frac{\ddot h_+}{a^2} - \frac{\dot a}{a^3}\dot h_+ + \frac{k^2 h_+}{a^2} = 2\frac{\dot a}{a^3}\dot h_+ + \frac{\ddot h_+}{a^2} + \frac{k^2 h_+}{a^2}$$

$$\Rightarrow a^2\left[\delta G^1{}_1 - \delta G^2{}_2\right] = \ddot h_+ + 2\frac{\dot a}{a}\dot h_+ + k^2 h_+$$

Easy to show that $T^1{}_1 - T^2{}_2 = 0$ if $\Theta = \Theta(\mu)$

↳ terms Θ that source scalar perturbations do not source tensor perturbations

$\to a^2\left[\delta G^1{}_1 - \delta G^2{}_2\right] = 0 \to \ddot h_+ + 2\frac{\dot a}{a}\dot h_+ + k^2 h_+ = 0$

Taking $\delta G^1{}_2$ we can follow the exact same steps for $h_\times$

$$\Rightarrow \boxed{\ddot h_\alpha + 2\frac{\dot a}{a}\dot h_\alpha + k^2 h_\alpha = 0 \qquad \alpha = +,\times}$$

Einstein equations for tensor perturbations, 1st order

(Damped) wave equation: gravitational waves!

If we neglect expansion of Universe

$\ddot h_\alpha + k^2 h_\alpha = 0 \to h_\alpha \propto e^{\pm ik\eta} \to h_\alpha(\vec x, \eta) = \int d^3k\, e^{i\vec k\cdot\vec x}\left[A e^{ik\eta} + B e^{-ik\eta}\right]$

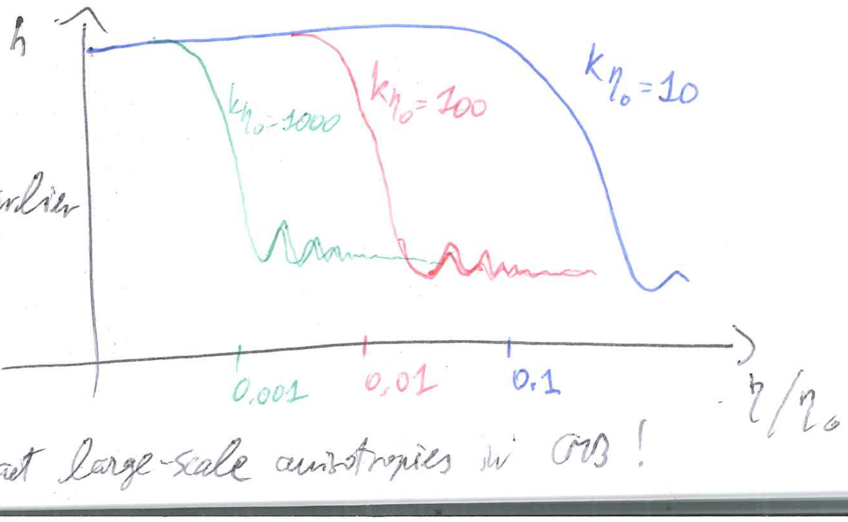
GWs travelling in $\pm\hat z$ direction at speed of light

For the full equation, solutions are damped oscillations. Modes start being damped when their wavelengths enter the horizon: $k\eta \approx 1$

$k\eta \ll 1 \to h \sim \text{const}$

$k\eta \gg 1 \to h$ damped

Small-scale modes decay earlier than large-scale modes



Decoupling: $\frac{\eta}{\eta_0} \simeq 0.02$, only modes with $k\eta_0 \lesssim 100$ survive

↳ GWs only impact large-scale anisotropies in CMB!