Horizon and flatness problems

Previously we set initial conditions at η_i such that

$$k\eta \ll 1 \quad \text{for all } k \text{ of } \underbrace{\text{interest}}_{\text{cosmological interest}}$$

$k \sim \frac{1}{\lambda}$ so $k\eta \sim \frac{\eta}{\lambda}$

$\eta \rightarrow$ comoving horizon

$\lambda \rightarrow$ ~~wavelength~~ comoving wavelength of perturbation

$k\eta \ll 1 \Longrightarrow \lambda \gg \eta$

Meaning of comoving horizon η and comoving Hubble radius $\frac{1}{aH}$

Comoving horizon η: objects separated by distances $> \eta$ could **NEVER** have communicated / been in causal contact

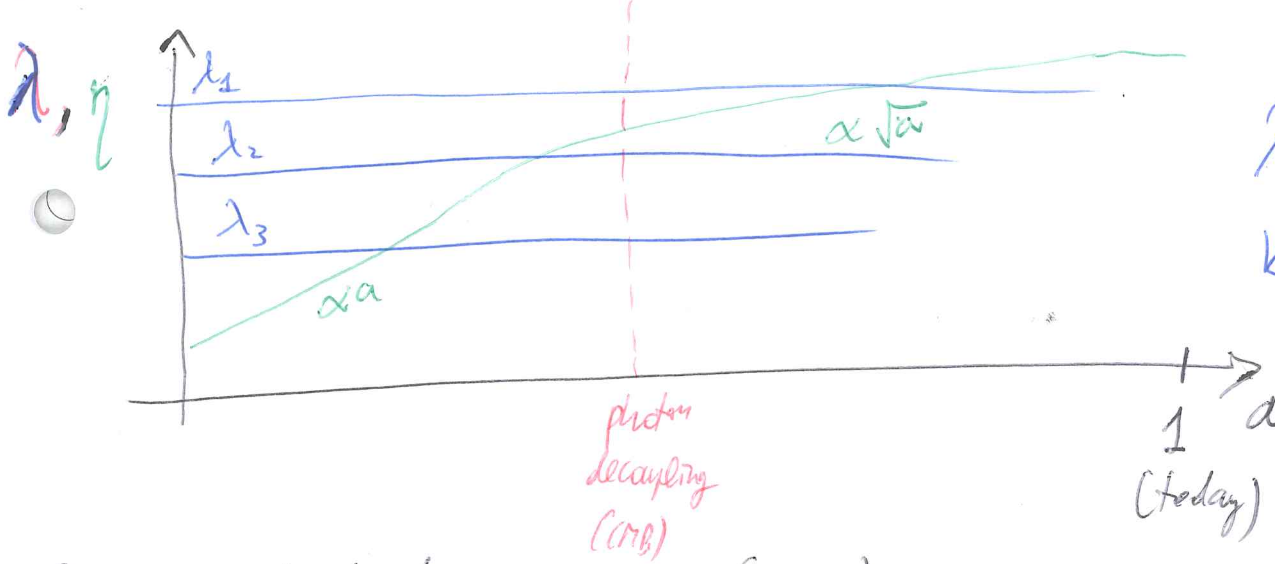
Comoving Hubble rate $\frac{1}{aH}$: object separated by distances $> \frac{1}{aH} = \frac{1}{\mathcal{H}}$ ~~co~~ cannot communicate now but might have been able to communicate in the past if $\eta > \frac{1}{aH}$

So $k\eta \gg 1$ or equivalently $\lambda \gg \eta$ means we are setting initial conditions at a time when the mode could not be affected by causal physics

Recall evolution of λ and η with scale factor

$\lambda = \text{const}$ (COMOVING wavelength)

$$\eta \begin{cases} \propto a^{1} & \text{RD} \\ \propto a^{\frac{1}{2}} & \text{MD} \end{cases} \quad \text{use } d\eta = \frac{dt}{a} \text{ with } \begin{array}{l} a \propto t^{1/2} \text{ (RD)} \\ a \propto t^{2/3} \text{ (MD)} \end{array}$$



$\lambda_1 > \lambda_2 > \lambda_3$

$k_1 < k_2 < k_3$

Since in standard cosmologies (MD, RD) η grows monotonically, all modes of cosmological interest were once outside the horizon, and only later they entered the horizon and causal physics could start to act on them

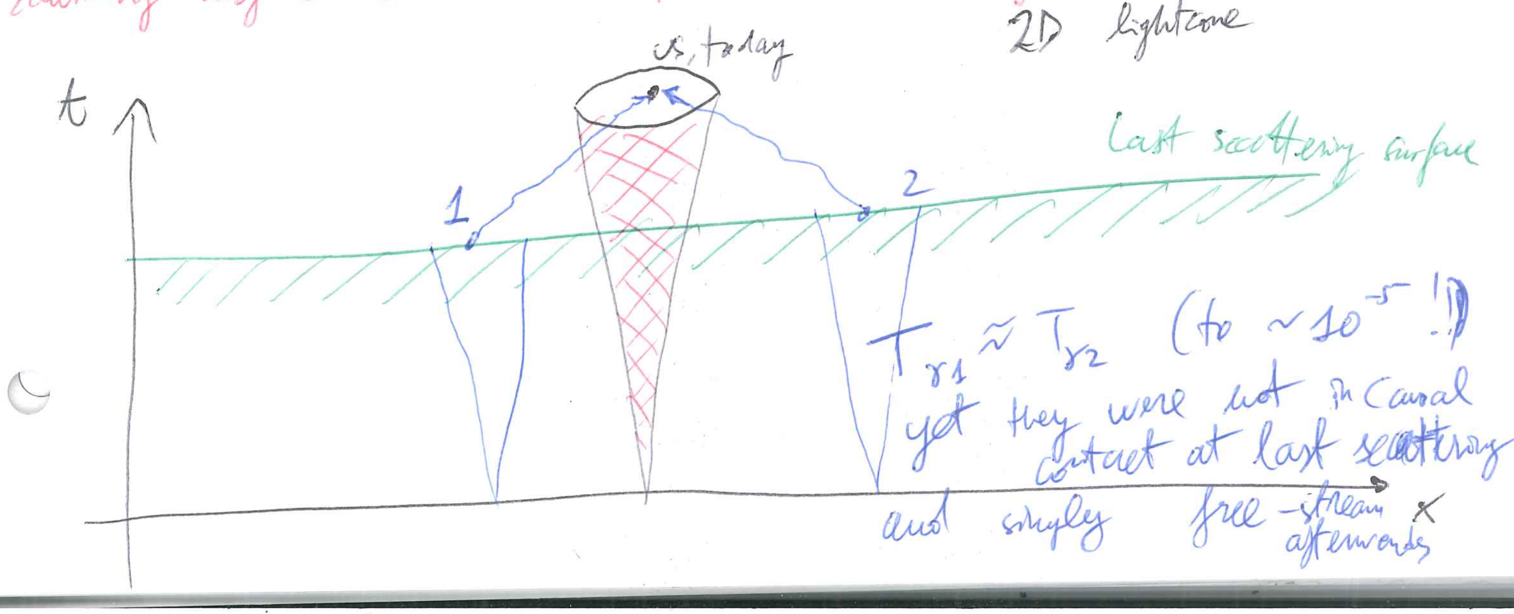
Problem: looking at the CMB, on all scales it is nearly isotropic, even if the longest scales we are observing entered the horizon only recently, well after photon decoupling when the CMB was formed! How did they "know" they "had to be" at similar temperatures across themselves if they contain regions which were never in causal contact?

(e.g. $\lambda_1 \sim$ CMB quadrupole scale)

↳ $\boxed{\text{HORIZON PROBLEM}}$

Why do regions separated by distances larger than the horizon at the last scattering surface share temperatures extremely close to each other?

2D lightcone



More mathematically:

Horizon size at last-scattering $z \sim 1100$, assume purely matter dominated (makes little difference...)

Angular diameter distance to last-scattering

$$d_A = \frac{\chi}{1+z} = \frac{1}{1+z}\int_0^z \frac{dz'}{H(z')} \simeq \frac{1}{1+z}\int_0^z \frac{dz'}{H_0\sqrt{\Omega_m(1+z')^3}} = \frac{2}{\sqrt{\Omega_m}H_0}\underbrace{\left(1-\frac{1}{\sqrt{1+z}}\right)}_{z\gg 1}\frac{1}{1+z}$$

$H(z) \simeq H_0\sqrt{\Omega_m(1+z)^3}$

$$\simeq \frac{2}{\sqrt{\Omega_m}H_0}\,\frac{1}{1+z}$$

$$d_A \simeq \frac{2}{\sqrt{\Omega_m}H_0(1+z)} \simeq O(10\ \mathrm{Mpc})$$

Horizon size at last-scattering

$$\eta_{LS} = \int_z^{\infty} \frac{dz'}{H(z')} = -\frac{2}{\sqrt{\Omega_m}H_0}\,\frac{1}{\sqrt{1+z}}\bigg|_z^{\infty} \simeq \frac{2}{\sqrt{\Omega_m}H_0\sqrt{1+z}} \simeq 0.3\ \mathrm{Mpc}$$

Today the physical size of this scale is

$$d_{LS} = (1+z)\eta_{LS} \simeq 1000 \times \eta_{LS} \simeq \cancel{3000}\ 300\ \mathrm{Mpc}$$

Angle subtended on the sky

$$\vartheta = \frac{d_{LS}}{d_H} \simeq \frac{300\ \mathrm{Mpc}}{13\ \mathrm{Gpc}} \boxed{\simeq 2^\circ}$$

d_H ≈ size of Universe

On angular scales $\gtrsim 2^\circ$ we would not expect the CMB to be uniform, yet it is uniform across basically the whole sky! Yet the corresponding photons were not in causal contact when they were emitted

~~Different way of phrasing horizon problem:~~

Different way of phrasing horizon problem: in a standard cosmology the horizon grows with time, so at any given instant scales that were never in causal contact enter the horizon and come into contact
$\Rightarrow$ we would expect them to be very inhomogeneous especially if they entered very recently.

$\boxed{\text{FLATNESS PROBLEM}}$

Friedmann equation with curvature

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \Longrightarrow \frac{K}{a_0^2} = \frac{8\pi G}{3}\rho_0 - H_0^2 = H_0^2\left(\frac{\rho_0}{\underbrace{\frac{3H_0^2}{8\pi G}}_{\rho_c}} - 1\right) = H_0^2\left(\frac{\rho}{\rho_c}-1\right) = H_0^2(\Omega_0 - 1)$$

Define $\Omega \equiv \sum_i \Omega_i$ (excluding curvature)

$$\Rightarrow H^2 = H_0^2\left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + (1-\Omega)a^{-2}\right]$$

$$\longrightarrow \frac{K}{a_0^2} = H_0^2\left(\frac{\rho}{\rho_c}-1\right) = H_0^2(\Omega_0 - 1)$$

$$\Rightarrow \boxed{\Omega(t) - 1 = \Omega_0 - 1\left(\frac{a}{a_0}\right)^{-2}\left(\frac{H}{H_0}\right)^{-2}}$$

$(a<a_0)$ $\quad \frac{K}{a^2} = H^2(\Omega - 1)$

~~$\Omega_K(a) = -\frac{K}{a^2H^2(a)} \Rightarrow \Omega_K, H\Omega_K(1+3w)$~~

$$\Omega(t) - 1 = \frac{\Omega_0 - 1}{\left(\frac{a}{a_0}\right)^2\left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + (1-\Omega)a^{-2}\right]} =$$

$$\underset{(a_0=1)}{=} \frac{\Omega_0 - 1}{\left[\Omega_m a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2 + (1-\Omega)\right]}$$

$\Omega(t) \to 1 \ \forall \Omega_0$ $\quad$ $a\to 0$, $z \to \infty$

$a \to 1$, $z\to 0$: Any small deviation from $\Omega(t)=1$ in the past is ENORMOUSLY magnified today

Today we measure ~~Ω_0~~ $\rho \sim \rho_c$ $(\Omega - 1)$ to $< 1\%$

$\Omega = 1$ is an unstable fixed point $\quad \Leftarrow \quad$ Ω had to be extremely close to 1 in the past! Why???

Inflation

Inflation solves the horizon and flatness problems and provides a mechanism for generating primordial perturbations

→ Early epoch of rapid (exponential) expansion sourced by negative pressure

How to solve ~~the~~ horizon problem?

Look at how η evolves in the past

$$d\eta = \frac{dt}{a} \rightarrow \eta = \int_0^{\eta} d\eta' = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{dt'}{da'} \frac{da'}{a'} = \int_0^a \frac{da'}{a'} \left(\frac{da'}{dt'} \right)^{-1} =$$

$$= \int_0^a \frac{da'}{a'} \frac{1}{a'} \Big(\underbrace{\frac{1}{a'} \frac{da'}{dt'}}_{H(a')} \Big)^{-1} = \int_0^a \frac{da'}{a'} \frac{1}{a' H(a')} = \int_0^a d(\ln a') \frac{1}{a' H(a')}$$

η: logarithmic integral of comoving Hubble horizon $\frac{1}{aH}$

How do η and $\frac{1}{aH}$ evolve?

	MD	RD
η	$\propto a^{1/2} \propto t^{1/3}$	$\propto a \propto t^{1/2}$
$\frac{1}{aH}$	$\propto t^{1/3}$	$\propto t^{1/2}$
a	$\propto t^{2/3}$	$\propto t^{1/2}$

Could it be that on some very large scale d

$$\frac{1}{aH} < d < \eta$$

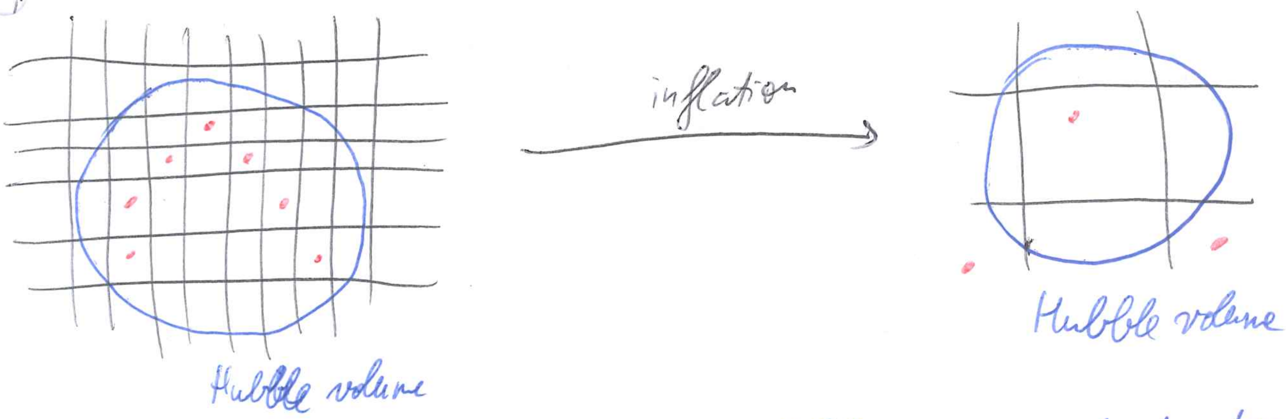
objects cannot communicate today...

...yet they could have communicated in the past

$$\eta = \int_0^a d(\ln a') \frac{1}{a' H(a')}$$

Since in standard cosmology $\frac{1}{\mathcal{H}} = \frac{1}{aH} \propto t^n \quad n>0$, η always picks up most of its contributions from the latest times. But if η got most of its contributions from early times, this could solve the horizon problem so $\eta \gg \frac{1}{aH}$

What if $\frac{1}{aH}$ decreased in the past? η would get most of its contributions from early times



: Hubble volume / physical Hubble radius ~ constant during inflation

How to make $(aH)^{-1}$ decrease $\longrightarrow$ aH increase?

$$\frac{d}{dt}(aH) = \frac{d}{dt}\left(a \frac{da/dt}{a}\right) = \frac{d^2a}{dt^2} > 0 \longrightarrow \text{accelerating period!}$$

Inflation

Recall

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) > 0 \longrightarrow \rho + 3P < 0 \longrightarrow \frac{P}{\rho} = w < -\frac{1}{3}$$

NEGATIVE PRESSURE

Typically $H \sim$ const during inflation $\frac{1}{a}\frac{da}{dt} = H = \text{const} \rightarrow \int \frac{da}{a} = H \int dt \rightarrow \ln a \propto Ht$

$\rightarrow a(t) \propto e^{Ht}$

t_e : end of inflation (time), scale factor a_e

$$a(t) = a_e e^{H(t - t_e)} \qquad (t < t_e)$$

$a(t) \propto e^{Ht}$ $\quad H$ const

$\rightarrow \frac{1}{aH} \propto \frac{1}{e^{Ht} H} \propto e^{-Ht}$ decrease in comoving Hubble horizon entirely due to exponential expansion

"How much" inflation do we need?

Typically inflation operates at temperatures $\sim 10^{15}$ GeV

Assume only radiation domination

$H \propto a^{-2} \longrightarrow aH \propto a^{-1} \longrightarrow \frac{a_0 H_0}{a_e H_e} = \frac{a_e}{a_0} = a_e$ end of inflation

$T_0 \sim 10^{-13}$ GeV $\sim 10^{-4}$ eV

$a_e \simeq \frac{T_0}{10^{15}\text{GeV}} \simeq 10^{-28}$

$\frac{a_0 H_0}{a_e H_e} \simeq 10^{-28}$ $\longrightarrow$ comoving Hubble radius at the end of inflation ~ 28 orders of magnitude smaller than today

Comoving Hubble radius at start of inflation has to be at least as large as largest observable scales today $\sim \frac{1}{a_0 H_0}$

$\longrightarrow$ During inflation comoving Hubble radius has to decrease by at least 28 orders of magnitude

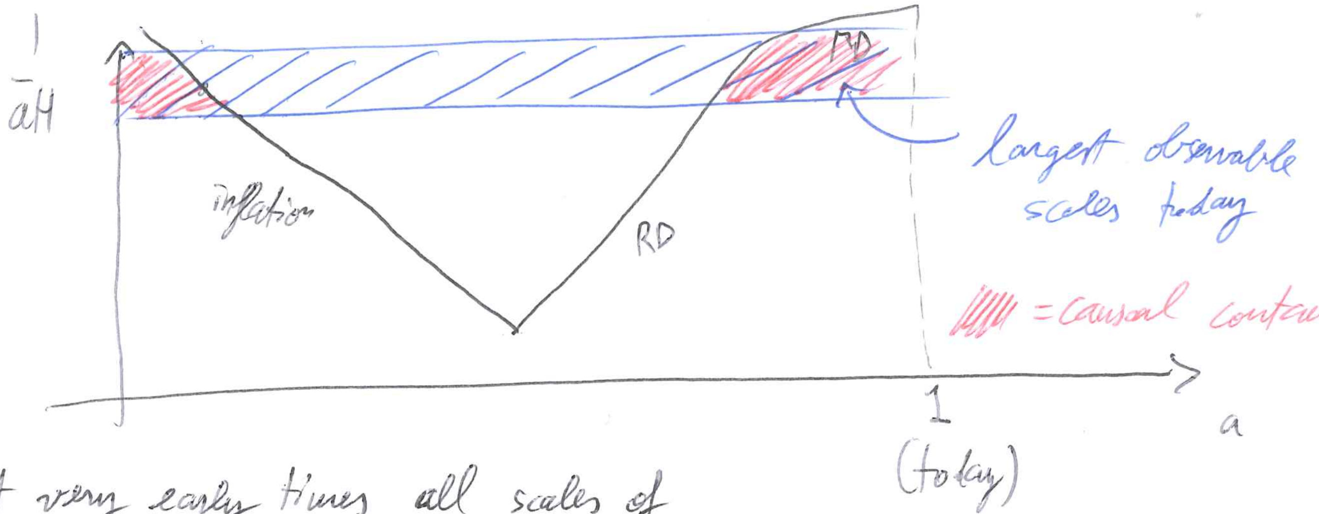
$\frac{1}{aH} \propto \cancel{e^{Ht}}\ \frac{1}{a} \propto e^{-Ht}$

argument of exponential has to ~~increase~~ change by $\ln(10^{28}) \sim O(60)$

$\hookrightarrow$ To solve horizon problem Universe needs at least 60 e-folds of inflation!

Flatness problem: exponential expansion makes space "very flat"

In practice $\Omega_k = 0$ becomes a stable attractor



At very early times all scales of cosmological interest were sub-horizon: Causal microphysics can operate!

During inflation they exit the horizon, and then re-enter

Note: in physical coordinates inflation blows up the physical size of a causally connected region $\Longrightarrow$ regions of cosmological size today were microscopically small during inflation (quantum?)

η gets huge contribution in the very early Universe, then it increases more slowly: no longer an effective time variable

Redefine it

$$\eta \rightarrow \eta - \underbrace{\eta_{prim}}_{\text{primordial part}}$$

$$\eta = \int_{t_e}^{t} \frac{dt'}{a(t')} \qquad \cancel{\left[\eta_{prim} = \int_{0}^{t_e} \frac{dt'}{a(t')}\right]}$$

↳ according to this definition $\eta < 0$ during inflation $(t < t_e)$ but always monotonically increasing

scale leaves horizon when $k|\eta| \lesssim 1$, re-enters horizon when $k|\eta| \gtrsim 1$

Before and during inflation the perturbations/modes were somehow given their initial conditions

Recall we need $\quad P < -\frac{\rho}{3} \Longrightarrow P < 0 \quad$ (like dark energy)

How to get $P<0$? Ordinary matter $(P=0)$ and radiation $(P=\frac{\rho}{3})$ won't do!