

From the decomposition theorem

Perturbations to the metric: scalar, vector, tensor \rightarrow each evolves independently (do not induce each other)

Recall we considered scalar perturbations from

$$G^0_0 \quad (\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij}) G^i_j$$

To show that the equations governing scalars are not affected by tensors, we need to show that tensor perturbations do not contribute to these two components

\bullet G^0_0 depends on R^0_0, R . We have already seen that

$$(\delta R^0_0)_{\text{tensor 1st order}} = 0 \quad \text{and} \quad (\delta R)_{\text{tensor 1st order}} = 0$$

\bullet $(\hat{k}_i \hat{k}_j - \frac{\delta_{ij}}{3}) G^i_j$

$$\delta G^i_j = \delta^{ik} \left(\frac{3}{2} H \frac{\partial \mathcal{H}_{kj}}{\partial t} + \frac{1}{2} \frac{\partial^2 \mathcal{H}_{kj}}{\partial t^2} + \frac{k^2}{2a^2} \mathcal{H}_{kj} \right)$$

$$(\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij}) \delta G^i_j = \left(\delta_{i3} \delta_{j3} - \frac{1}{3} \delta_{ij} \right) \left(\frac{3}{2} H \frac{\partial \mathcal{H}_{ij}}{\partial t} + \frac{1}{2} \frac{\partial^2 \mathcal{H}_{ij}}{\partial t^2} + \frac{k^2}{2a^2} \mathcal{H}_{ij} \right)$$

$\delta_{i3} \delta_{j3} \delta_{i3} \delta_{j3} (\dots) \propto \frac{\partial \mathcal{H}_{33}}{\partial t}, \frac{\partial^2 \mathcal{H}_{33}}{\partial t^2}, \mathcal{H}_{33}, \mathcal{H}_{33} = 0 \Rightarrow = 0$

$\delta_{ij} (\dots) \propto \frac{\partial \mathcal{H}}{\partial t}, \frac{\partial^2 \mathcal{H}}{\partial t^2}, \mathcal{H}, \mathcal{H} = \mathcal{H}_{ii} = 0 \Rightarrow = 0$

So scalar equations are unchanged by the presence of tensor modes!

One could make the same argument for vector modes

\downarrow
not considered often!

(need exotic models e.g. topological defects)
Just scalars and tensors

Gauge transformations

Back to scalar perturbations. So far we have worked in a specific gauge/coordinate system

$$ds^2 = -(1+2\psi)dt^2 + a^2\delta_{ij}(1+2\phi)dx^i dx^j$$

↳ conformal Newtonian gauge

Other well-studied gauges:

- spatially flat slicing (for inflation) g_{ij} unperturbed
 - synchronous ~~rest~~ gauge (perturbations better behaved) g_{00} unperturbed
- ↳ numerically

But observations are gauge-invariant, and it's useful to know how to move between gauges

General scalar perturbation to the metric

$$g_{0i} = -a \partial_i B$$

$$g_{ij} = a^2 \left(\delta_{ij} [1+2\psi] - 2 \frac{\partial^2 E}{\partial x_i \partial x_j} \right)$$

(derivative of a scalar function)

$$g_{0i} = -a \partial_i B$$

4 functions characterize scalar perturbations: A, B, ψ, E

Conformal Newtonian gauge: $A = \psi, B = 0, \psi = \phi, E = 0$

Transform between gauges imposing that invariant distance does not depend on the coordinates that we use

$$x \longrightarrow \tilde{x}$$

$$\tilde{g}_{\alpha\beta}(\tilde{x}) d\tilde{x}^\alpha d\tilde{x}^\beta = g_{\mu\nu}(x) dx^\mu dx^\nu$$

$$d\tilde{s}^2 = ds^2$$

$$\tilde{g}_{\alpha\beta} d\tilde{x}^\alpha d\tilde{x}^\beta = \tilde{g}_{\alpha\beta}(\tilde{x}) \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} dx^\mu dx^\nu = g_{\mu\nu}(x) dx^\mu dx^\nu$$

\uparrow $d\tilde{x}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} dx^\mu$ $d\tilde{x}^\beta = \frac{\partial \tilde{x}^\beta}{\partial x^\nu} dx^\nu$

$$\Rightarrow \tilde{g}_{\alpha\beta}(\tilde{x}) \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} = g_{\mu\nu}(x)$$

How metric changes under a coordinate transformation

Most general coordinate transformation

$$t \rightarrow \tilde{t} = t + \xi^0(t, \vec{x})$$

$$x^i \rightarrow \tilde{x}^i = x^i + \xi^i(t, \vec{x})$$

ξ^0, ξ^i small perturbations (same order as ψ, Φ etc.)

Let's work out one component explicitly :00 ($\mu=0, \nu=0$)

$$\tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^\alpha}{\partial x^0} \frac{\partial \tilde{x}^\beta}{\partial x^0} = \tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^\alpha}{\partial t} \frac{\partial \tilde{x}^\beta}{\partial t} = g_{00}(x) = -(1+2A)$$

$\tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^\alpha}{\partial t} \frac{\partial \tilde{x}^\beta}{\partial t}$ only $\alpha=\beta=0$ contributes

Example: $\alpha=0$ $\beta=i$

$$\tilde{g}_{0i} \frac{\partial \tilde{x}^0}{\partial t} \frac{\partial \tilde{x}^i}{\partial t} \propto \frac{\partial B}{\partial x^i} \frac{\partial \xi}{\partial t^2}$$

2nd order

$$\propto \frac{\partial B}{\partial x^i} \frac{\partial \xi}{\partial t^2}$$

similarly $\alpha=i$ $\beta=j$

$$\tilde{g}_{\alpha\beta} \frac{\partial \tilde{x}^\alpha}{\partial t} \frac{\partial \tilde{x}^\beta}{\partial t} \approx \tilde{g}_{00} \frac{\partial \tilde{x}^0}{\partial t} \frac{\partial \tilde{x}^0}{\partial t} = \tilde{g}_{00} \left(\frac{\partial \tilde{t}}{\partial t} \right)^2 = -(1+2A) \left(\frac{\partial \tilde{t}}{\partial t} \right)^2 =$$

$$\tilde{t} = t + \xi^0$$

$$\frac{\partial \tilde{t}}{\partial t} = 1 + \frac{\partial \xi^0}{\partial t}$$

$$= -(1+2\tilde{A})\left(1+\frac{\partial\zeta^0}{\partial t}\right)^2 \underset{\zeta^0 \ll 1}{\approx} -(1+2\tilde{A})\left(1+2\frac{\partial\zeta^0}{\partial t}\right) \approx -1-\tilde{A}-2\frac{\partial\zeta^0}{\partial t}$$

$\frac{\partial\zeta^0}{\partial t}$ 2nd order

$$\approx g_{00} = -1-2A$$

$$-\cancel{1}-2\tilde{A}-2\frac{\partial\zeta^0}{\partial t} = -1-2A \implies -2\tilde{A}-2\frac{\partial\zeta^0}{\partial t} = -2A \implies \tilde{A} = A + \frac{\partial\zeta^0}{\partial t}$$

so under coordinate transformation

$$t \rightarrow \tilde{t} = t + \zeta^0(t, \vec{x})$$

$$x^i \rightarrow \tilde{x}^i = x^i + \int^{t+\zeta^0} \frac{\partial\zeta^i}{\partial x^j}(t, \vec{x}) dx^j$$

A transforms as

$$A \rightarrow \tilde{A} = A - \frac{\partial\zeta^0}{\partial t} = A - \frac{\dot{\zeta}^0}{a}$$

$\frac{d}{dt} = \frac{dt}{a} \frac{d}{dt} \implies \frac{d}{dt} = \frac{1}{a} \frac{d}{dt}$

Similarly find how other components transform

$$\tilde{\Psi} = \Psi - H\zeta^0$$

$$\tilde{B} = B - \frac{\dot{\zeta}^0}{a} + \dot{\zeta}^0$$

$$\tilde{E} = E + \dot{\zeta}^0$$

Note: these transformations describe how metric tensor changes, the corresponding scalars (e.g. A) don't change in a proper sense, the apparent change is just a redefinition of these scalars!

Although there are 4 functions characterizing scalar perturbations, we can use Gauge invariance to eliminate 2 of them

Example: start with metric where $E \neq 0$

$$\rightarrow \text{Choose } \zeta^0 = -E \rightarrow \tilde{E} = E - \dot{\zeta}^0 = 0$$

So really only $4-2=2$ functions are needed to describe scalar perturbations! (e.g. Φ, Ψ)

In fact can construct gauge invariant variables which do not change under gauge transformations. Bardeen variables

$$\bar{\Phi}_A \equiv A + \frac{1}{a} \frac{\partial}{\partial \eta} [a(\dot{E} - B)] \longrightarrow \Psi$$

$$\bar{\Phi}_H \equiv -\Psi + aH(B - \dot{E}) \longrightarrow -\Phi$$

\uparrow
 conformal Newtonian gauge
 $\dot{E} = B = 0$

Useful if we want to calculate perturbations in a gauge where equations become very simple

$T_{\mu\nu}$ also changes under gauge transformations

$$\tilde{T}_{\mu\nu}(\vec{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} T_{\alpha\beta}(x) \longrightarrow 2 \text{ combinations of components of } T_{\mu\nu} \text{ are gauge-invariant}$$

Fourier space

$$v \equiv i k B + \frac{\dot{v}}{k} \frac{T^0_i}{(p+p)a}$$

$$E_m \equiv -1 - \frac{T^0_0}{\rho} + \frac{3H}{k^2 \rho} k^i T^0_i$$

← Conformal Newtonian gauge →

$$v = \begin{cases} v_b & b \\ v & DM \\ -3i\theta_1 & \gamma \\ 3i\theta_1 \sim "v_y" & \end{cases}$$

goes to $\delta_{ij} \propto$
(standard overdensity)
on small scales, $k \rightarrow \infty$

$$E = \begin{pmatrix} \delta_b + \frac{3aHv_b}{k} & b \\ \delta + \frac{3aHv}{k} & DM \\ 4\theta_1 - 42i\theta_1 \frac{aH}{k} & \end{pmatrix}$$

Summary of Einstein equations

$$\begin{cases} k^2 \bar{\Phi} + 3 \frac{\dot{a}}{a} (\dot{\bar{\Phi}} - \Psi \frac{\dot{a}}{a}) = 4\pi G a^2 (\rho_{dm} \delta + \rho_b \delta_b + 4\rho_\gamma \theta_0 + 4\rho_\nu N_0) & [\delta G^0_0] \\ k^2 (\bar{\Phi} + \Psi) = -32\pi G a^2 (\rho_\gamma \theta_2 + \rho_\nu N_2) & [(k_i k^i - \frac{1}{3} \delta^j_j) G^i_j] \end{cases}$$

Scalar perturbations

$$\rightarrow k^2 \bar{\Phi} = 4\pi G a^2 \left[\rho_{dm} \delta + \rho_b \delta_b + 4\rho_\gamma \theta_0 + 4\rho_\nu N_0 + \frac{3aH}{k} (\rho_{dm} \delta + \rho_b \delta_b + 4\rho_\gamma \theta_0 + 4\rho_\nu N_0) \right]$$

useful combination!

$$\begin{cases} \ddot{h}_+ + 2 \frac{\dot{a}}{a} \dot{h}_+ + k^2 h_+ = 0 \\ \ddot{h}_\times + 2 \frac{\dot{a}}{a} \dot{h}_\times + k^2 h_\times = 0 \end{cases}$$

tensor perturbations

INITIAL CONDITIONS AND INFLATION

- Full Einstein-Boltzmann system for perturbations to the particle distributions and the metric (recap)

$$\dot{\Theta} + ik_\mu \Theta = -\dot{\Phi} - ik_\mu \Psi - \dot{\tau} [\Theta_0 - \Theta + \mu v_b] \quad \left. \vphantom{\dot{\Theta} + ik_\mu \Theta} \right\} \gamma$$

$$\dot{\delta}_{dm} + ik_\mu v_{dm} = -3\dot{\Phi} \quad \left. \vphantom{\dot{\delta}_{dm} + ik_\mu v_{dm}} \right\} DM$$

$$\dot{v} + \frac{\dot{a}}{a} v = -ik \Psi$$

$$\dot{\delta}_b + \frac{\dot{a}}{a} \delta_b + ik v_b = -3\dot{\Phi} \quad \left. \vphantom{\dot{\delta}_b + \frac{\dot{a}}{a} \delta_b} \right\} b$$

$$\dot{v}_b + \frac{\dot{a}}{a} v_b = -ik \Psi + \frac{\dot{\tau}}{R} [v_b + 3i\theta_1]$$

$$\dot{N} + ik_\mu N = -\dot{\Phi} - ik_\mu \Psi \quad \left. \vphantom{\dot{N} + ik_\mu N} \right\} \nu$$

$$K^2 \Phi + 3 \frac{\dot{a}}{a} \left(\dot{\Phi} - \Psi \frac{\dot{a}}{a} \right) = 4\pi G a^2 (\rho_{dm} \delta + \rho_b \delta_b + 4\rho_s \theta_0 + 4\rho_\nu N_0) \quad \left. \vphantom{K^2 \Phi + 3 \frac{\dot{a}}{a} \left(\dot{\Phi} - \Psi \frac{\dot{a}}{a} \right)} \right\} \begin{array}{l} \text{scalar} \\ \text{perturbations} \\ \Phi, \Psi \end{array}$$

$$K^2 (\dot{\Phi} + \Psi) = -32\pi G a^2 (\rho_s \theta_2 + \rho_\nu N_2)$$

$$\ddot{h}_+ + 2 \frac{\dot{a}}{a} \dot{h}_+ + K^2 h_+ = 0 \quad \left. \vphantom{\ddot{h}_+ + 2 \frac{\dot{a}}{a} \dot{h}_+} \right\} \begin{array}{l} \text{tensor} \\ \text{perturbations} \\ h_+, h_x \end{array}$$

$$\ddot{h}_x + 2 \frac{\dot{a}}{a} \dot{h}_x + K^2 h_x = 0$$

Neglected polarization, assuming massless neutrinos, to 1st order

$$4\theta_0 \sim \delta_\gamma \quad 4N_0 \sim \delta_\nu \quad 3i\theta_1 \sim v_\gamma \quad -3iN_1 \sim v_\nu$$

But what are the initial conditions for $\Theta, \delta, v, N, \Phi, \Psi, h$?

Inflation can explain why the CMB sky is so uniform and

provide the initial conditions for the perturbations

⤴ Caution: we don't know for sure whether inflation really took place.

The Einstein-Boltzmann equations at early times

It turns out we only really need initial conditions for Φ , then we can relate all other variables to Φ

Consider Boltzmann equations for $\Theta, \delta_{dm}, \delta_b, v_{dm}, v_b, N$ at very early times: $k\eta \ll 1$ for any k mode we care about
 Look at photon equation

$$\dot{\Theta} + ik_\mu \Theta = -\dot{\Phi} - ik_\mu \Psi - \dot{\tau} [\Theta_0 - \Theta + \mu v_b]$$

$$\begin{aligned} \textcircled{1} &\sim \frac{\Theta}{\tau} \quad \rightarrow \quad \frac{\textcircled{1}}{\tau} \sim \frac{\Theta}{k\eta} \sim \frac{1}{k\eta} \gg 1 \text{ by assumption!} \\ \textcircled{2} &\sim k\Theta \end{aligned}$$

Similarly can argue that all terms multiplied by k can be neglected at early times: wavelengths $\sim k^{-1} \gg$ distance over which causal physics operates

\hookrightarrow observer ~~there~~ sees only photons from causal horizon with nearly uniform temperature

\hookrightarrow only monopole survives $\Theta_0 \gg \Theta_1, \Theta_2, \dots$

$$\dot{\Theta} + ik_\mu \Theta = -\dot{\Phi} - ik_\mu \Psi - \dot{\tau} [\Theta_0 - \Theta + \mu v_b]$$

$\Theta_0 - \Theta$
 $v_b \sim$ 1st order moment ~ 0

$$\dot{N} + ik_\mu N = -\dot{\Phi} - ik_\mu \Psi$$

$$\begin{cases} \dot{\Theta}_0 + \dot{\Phi} = 0 \\ \dot{N}_0 + \dot{\Phi} = 0 \end{cases}$$

$\Theta \sim \Theta_0$
 $N \sim N_0$

$$\begin{aligned} \dot{v}_b + \frac{\dot{a}}{a} v_b &= -ik_\mu \Psi + \frac{\dot{\tau}}{R} [v_b + 3\Theta_1] \\ \dot{\tau} \text{ huge} &\rightarrow v_b \approx -3\Theta_1 \end{aligned}$$

Set $N \sim 0$

$N_b \sim 0$

since the v s are basically 1st order moments
 Even for matter only monopole counts

~~$\delta_a + i k v_{dm} = -3\dot{\Phi}$~~

$$\begin{cases} \delta_a + i k v_{dm} = -3\dot{\Phi} \\ \delta_b + i k v_b = -3\dot{\Phi} \end{cases} \longrightarrow \begin{cases} \dot{\delta} = -3\dot{\Phi} \\ \dot{\delta}_b = -3\dot{\Phi} \end{cases}$$

Now consider Einstein equations at early times

$$k^2 \dot{\Phi} + 3 \frac{\dot{a}}{a} \left(\dot{\Phi} - \psi \frac{\dot{a}}{a} \right) = 4\pi G a^2 (\rho_m \delta + \rho_r \delta + 4\rho_r \theta_0 + 4\rho_\nu N_0)$$

Radiation is dominating

$$\longrightarrow 3 \frac{\dot{a}}{a} \left(\dot{\Phi} - \psi \frac{\dot{a}}{a} \right) = 16\pi G a^2 (\rho_r \theta_0 + \rho_\nu N_0)$$

During radiation domination $a \propto t^{1/2}$

$$\int \frac{dt}{a} = \frac{dt}{a} \propto \int \frac{dt}{t^{1/2}} \sim t^{1/2} \quad \eta \propto t^{1/2} \propto a \quad a \propto \eta \quad \frac{a}{c} = \eta \quad \frac{a}{a} = \frac{1}{\eta}$$

$$\frac{\dot{a}}{a} = \frac{1}{a} \frac{da}{dt} = \frac{1}{a} \frac{da}{d\eta} \frac{d\eta}{dt} = \frac{1}{\eta} \quad \frac{\dot{a}}{a} = \frac{1}{a} \frac{da}{d\eta} = \frac{1}{\eta}$$

$$\longrightarrow 3 \frac{1}{\eta} \left(\dot{\Phi} - \frac{\psi}{\eta} \right) = 16\pi G a^2 (\rho_r \theta_0 + \rho_\nu N_0) \longrightarrow \frac{\dot{\Phi}}{\eta} - \frac{\psi}{\eta^2} = \frac{16\pi G a^2}{3} \left(\frac{\rho_r}{\rho} \theta_0 + \frac{\rho_\nu}{\rho} N_0 \right)$$

$\frac{16\pi G a^2 \rho}{3}$

Friedmann equation

$$H^2 = \frac{1}{a^2} \left(\frac{da}{dt} \right)^2 = \frac{8\pi G}{3} \rho \longrightarrow \left(\frac{\dot{a}}{a} \right)^2 = a^2 H^2 = \frac{8\pi G \rho a^2}{3} = \frac{1}{\eta^2}$$

$$\longrightarrow \frac{16\pi G \rho a^2}{3} = \frac{2}{\eta^2}$$

$$* = \frac{2}{\eta^2} \left(\frac{\rho_r}{\rho} \theta_0 + \frac{\rho_\nu}{\rho} N_0 \right) = \frac{2}{\eta^2} \left(f_\nu \equiv \frac{\rho_\nu}{\rho_r + \rho_\nu} \right) \quad \rho_r = 1 - f_\nu$$

$$= \frac{2}{\eta^2} [(1 - f_\nu) \theta_0 + f_\nu N_0]$$

$$\longrightarrow \frac{\dot{\Phi}}{\eta} - \frac{\psi}{\eta^2} = \frac{2}{\eta^2} [(1 - f_\nu) \theta_0 + f_\nu N_0]$$

$$\longrightarrow \dot{\Phi} \eta - \psi = 2 [(1 - f_\nu) \theta_0 + f_\nu N_0]$$

But we also know that

$$\dot{\Theta}_0 + \dot{\Phi} = 0$$

$$\dot{N}_0 + \dot{\Phi} = 0$$

$$\dot{\Phi} \eta - \psi = 2([1-f_v] \Theta_0 + f_v N_0)$$

Differentiate both sides

$$\ddot{\Phi} \eta + \dot{\Phi} - \dot{\psi} = 2([1-f_v] \dot{\Theta}_0 + f_v \dot{N}_0)$$

$$\Downarrow \dot{\Theta}_0 = -\dot{\Phi}, \dot{N}_0 = -\dot{\Phi}$$

$$(1-f_v) \dot{\Theta}_0 + f_v \dot{N}_0 = -\dot{\Phi} + f_v \dot{\Phi} - f_v \dot{\Phi} = -\dot{\Phi}$$

$$\ddot{\Phi} \eta + \dot{\Phi} - \dot{\psi} = -2\dot{\Phi}$$

Look at the other Einstein equation

$$K^2(\Phi + \psi) = -32\pi G a^2 (\rho_2 \Theta_2 + p_2 N_2) \Rightarrow \psi \approx -\Phi$$

neglect higher-order moments

$$\ddot{\Phi} \eta + \dot{\Phi} + \dot{\Phi} = -2\dot{\Phi}$$

$$\ddot{\Phi} \eta + 4\dot{\Phi} = 0$$

Solve by $\Phi = \eta^p \Rightarrow \ddot{\Phi} = p(p-1)\eta^{p-2} \quad \dot{\Phi} = p\eta^{p-1}$

$$\Rightarrow p(p-1)\eta^{p-1} + 4p\eta^{p-1} = 0 \Rightarrow p(p-1) + 4p = 0$$

$$p^2 + 3p = p(p+3) = 0 \Rightarrow p = 0, p = -3$$

growing mode

which we care about

decaying mode cosmologically irrelevant!

Therefore we need to extract the growing mode

$$\Phi = \eta^0 = \text{const} \quad \dot{\Phi} = 0 \quad \psi = -\Phi$$

$$\Phi = 2([1-f_v] \Theta_0 + f_v N_0)$$

$$\hookrightarrow \Theta_0, N_0 = \text{const}$$

In most models of structure formation not only are Θ_0 and N_0 constant but also equal

$$\Theta_0(k, \eta_i) = N_0(k, \eta_i)$$

$$\Rightarrow \bar{\Phi} = 2[(1-f_0)\Theta_0 + f_0 N_0] \stackrel{\Theta_0=N_0}{=} 2\Theta_0$$

$$\hookrightarrow \boxed{\bar{\Phi}(k, \eta_i) = 2\Theta_0(k, \eta_i)}$$

Note:
 - explicit k -dependence
 - initial conditions set at very early η_i

Back to

$$\begin{aligned} \dot{\delta} &= -3\dot{\bar{\Phi}} & \dot{\Theta}_0 + \dot{\bar{\Phi}} &= 0 \Rightarrow & \dot{\delta} &= 3\Theta_0 + \text{const} \\ \dot{\delta}_b &= -3\dot{\bar{\Phi}} & & & \dot{\delta}_b &= 3\Theta_0 + \text{const} \end{aligned}$$

Two types of primordial perturbations

const = 0 \rightarrow ADIABATIC

const $\neq 0$ \rightarrow ISOCURVATURE \rightarrow severely constrained!

Adiabatic perturbations: constant number density ratio between all species everywhere

$$\frac{n_{dm}}{n_\gamma} = \frac{n_{dm}^{(0)}}{n_\gamma^{(0)}} \begin{bmatrix} 1 + \delta \\ 1 + 3\Theta_0 \end{bmatrix}$$

constant in space and time

$$\approx 1 + \delta - 3\Theta_0 = 1 \Rightarrow \delta = 3\Theta_0$$

to be independent of space and time

So $\delta = 3\Theta_0$ and similarly $\delta_b = 3\Theta_0$

Initial conditions for velocities (exercise)

$$\Theta_1 = N_1 = \frac{iN_b}{3} = \frac{i\sigma}{3} = -\frac{k\bar{\Phi}}{6aH}$$

Summarizing IC: $\bar{\Phi} = -\psi = 2\Theta_0$ $\delta = \delta_b = 3\Theta_0$ $\Theta_1 = N_1 = \frac{iN_b}{3} = \frac{i\sigma}{3} = -\frac{k\bar{\Phi}}{6aH}$