Scalar field implementation of inflation

Can a scalar field $\phi(\bar{x},t)$ give $P<0$?

$$T^{\alpha}{}_{\beta} = g^{\alpha\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^\beta} - g^{\alpha}{}_{\beta}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} + V(\phi)\right]$$

$(-+++)$ signature

$\left[V(\phi)=\frac{1}{2}m^2\phi^2 \text{ for free field}\right]$

$$\phi = \underbrace{\phi^0(t)}_{\text{homogeneous part}} + \underbrace{\delta\phi(\bar{x},t)}_{\text{1st order perturbation}}$$

Let's first just look at the homogeneous part to see how it affects the evolution of the scale factor

$$\phi \approx \phi^0(t) \longrightarrow \frac{\partial\phi}{\partial x^i}=0$$

$\cancel{T^{(0)\alpha}{}_{\beta} = -g^{\alpha}}$ $\qquad T^{(0)0}{}_{0} = -\rho \qquad T^{(0)i}{}_{i} = P$

$$T^{(0)0}{}_{0} = g^{0\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^0} - g^{0}{}_{0}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} + V(\phi)\right]$$

$\nu=0$ $\qquad = g^{00}\left(\frac{\partial\phi}{\partial x^0}\right)^2 - g^{0}{}_{0}\left[\frac{1}{2}g^{00}\left(\frac{\partial\phi}{\partial x^0}\right)^2 + V(\phi)\right] =$

$\mu=\nu=0$

$x^\nu=-t$

$g^{00}=-1,\ g^{0}{}_{0}=1$

$$= -\left(\frac{d\phi^{(0)}}{dt}\right)^2 - \left[-\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)\right] = -\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi) = -\rho$$

$$\Longrightarrow \rho = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)$$

$$T^{(0)i}{}_{i} = \cancelto{0}{g^{i}{}_{\nu}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^\nu}} - g^{i}{}_{i}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} + V(\phi)\right] = -g^{i}{}_{i}\left[-\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)\right] =$$

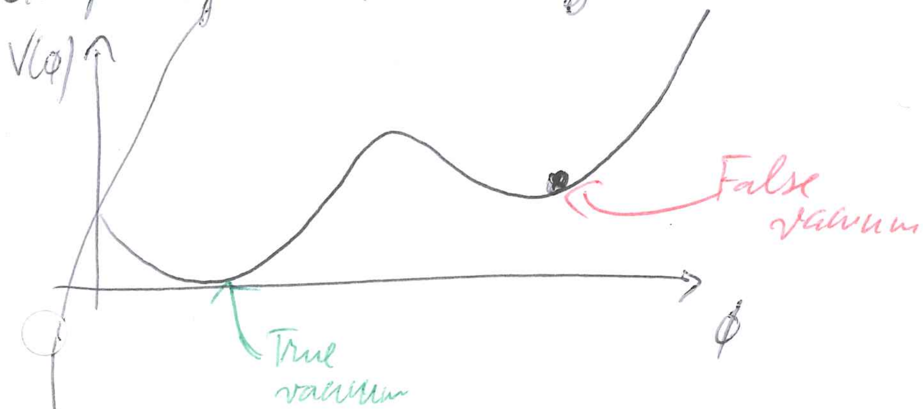
$g^{i}{}_{i}=1 \longrightarrow$ $= \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi) = P \longrightarrow P = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)$

$$\rho = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)$$

$$P = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi)$$

Field configuration with more potential than kinetic energy: negative pressure

Example: field trapped in false vacuum



$\phi^{(0)} \approx$ const so $\rho \approx V(\phi)$, $P \approx -V(\phi)$ $\rightarrow w = -1$

$\rho \approx V(\phi) =$ const $\rightarrow$ very different from matter (a^{-3}), radiation (a^{-4})
behaves like Λ!

An universe with even a bit of false vacuum energy will quickly be dominated by it

Scale factor evolution

$$H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho = \text{const} \rightarrow \frac{1}{a}\frac{da}{dt} = H = \text{const}$$

$$\int \frac{da}{a} = H\int dt \qquad \ln a = Ht + \text{const} \rightarrow a \propto e^{Ht} \propto e^{\sqrt{\rho}\, t} \propto e^{\sqrt{V(\phi)}\, t} \propto e^{\sqrt{V_{FV}}\, t}$$

Primordial comoving horizon $\eta_{prim} = \int_{t_e}^{t_b} \frac{dt'}{a(t')}$

$t_b \rightarrow$ beginning of inflation

$t_e \rightarrow$ end of inflation

$$\eta_{prim} = \int_{t_e}^{t_b} \frac{dt}{a_e e^{H(t-t_b)}} = \frac{1}{a_e}\int_{t_e}^{t_b} \frac{dt}{e^{H(t-t_b)}} = -\frac{1}{Ha_e} e^{-H(t-t_b)}\Big|_{t_e}^{t_b} =$$

$$= -\frac{1}{Ha_e}\left(e^{-H(t_b-t_b)} - e^{-H(t_e-t_b)}\right) = \frac{1}{Ha_e}\left(e^{H(t_e-t_b)} - 1\right)$$

$$\eta_{prim} = \frac{1}{Ha_e}\left(e^{H(t_e-t_i)}-1\right) \quad \text{requires} \quad H(t_e-t_i) > 60$$

(more than 60 e-foldings of inflation)

Nowadays most popular models based on field rolling down its potential, otherwise ~~regions of~~ "bubbles" of true vacuum never coalesce in time

Consider generic equations for scale factor when $V(\phi) \neq$ const

Recall two Friedmann equations

$$H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho \simeq \frac{8\pi G}{3}\left[\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)\right]$$

$$\frac{1}{a}\frac{d^2a}{dt^2} + \frac{1}{2}\left(\frac{1}{a}\frac{da}{dt}\right)^2 = -4\pi G P \quad \Longrightarrow \quad \frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(3P+\rho)$$

Take derivative of 1st Friedmann equation

$$\frac{d}{dt}(H^2) = 2H\frac{dH}{dt} \qquad \frac{dH}{dt} = \frac{d}{dt}\left(\frac{da/dt}{a}\right) = \frac{d^2a/dt^2\, a - (da/dt)^2}{a^2} =$$

$$= \frac{1}{a}\frac{d^2a}{dt^2} - \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{1}{a}\frac{d^2a}{dt^2} - H^2$$

$$\frac{d}{dt}(H^2) = 2H\frac{dH}{dt} = 2\frac{da/dt}{a}\left[\frac{d^2a/dt^2}{a} - \left(\frac{da/dt}{a}\right)\right]^2$$

$$\frac{d}{dt}(H^2) = \frac{d}{dt}\left(\frac{8\pi G}{3}\rho\right) = \frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)\right]_{\times\frac{8\pi G}{3}} = \left[\underbrace{2\cdot\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)\left(\frac{d^2\phi^{(0)}}{dt^2}\right)}_{\text{chain rule}} + \frac{dV(\phi)}{dt}\right]_{\times\frac{8\pi G}{3}} =$$

$$= \frac{8\pi G}{3}\left[\left(\frac{d\phi^{(0)}}{dt}\right)\left(\frac{d^2\phi^{(0)}}{dt^2}\right) + \underbrace{\frac{dV(\phi)}{d\phi^{(0)}}\frac{d\phi^{(0)}}{dt}}_{\text{chain rule}}\right] = \frac{8\pi G}{3}\left[\left(\frac{d\phi^{(0)}}{dt}\right)\left(\frac{d^2\phi^{(0)}}{dt^2}\right) + V'\frac{d\phi^{(0)}}{dt}\right]$$

$V' = dV(\phi)/d\phi^{(0)}$

Putting everything together

$$2\frac{da/dt}{a}\left[\underbrace{\frac{d^2a/dt^2}{a}}_{\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho+3P)} - \underbrace{\left(\frac{da/dt}{a}\right)}_{\left(\frac{1}{a}\frac{da}{dt}\right)^2 = H^2 = \frac{8\pi G}{3}\rho}\right]^2 = \frac{8\pi G}{3}\left[\left(\frac{d\phi^{(0)}}{dt}\right)\left(\frac{d^2\phi^{(0)}}{dt^2}\right) + V'\frac{d\phi^{(0)}}{dt}\right]$$

LHS

$$\Rightarrow 2\frac{da/dt}{a}\left[-\frac{4\pi G}{3}(\rho+3P) - \frac{8\pi G}{3}\rho\right] = 8\pi G\frac{da/dt}{a}\left[-\frac{\rho}{3} - P - \frac{2\rho}{3}\right] =$$

$$= -8\pi G\frac{da/dt}{a}(\rho+P) = -8\pi G\frac{da/dt}{a}\left(\frac{d\phi^{(0)}}{dt}\right)^2 = -8\pi G H\left(\frac{d\phi^{(0)}}{dt}\right)^2$$

$\rho = \frac{1}{2}\dot\phi^2 + V(\phi)$
$P = \frac{1}{2}\dot\phi^2 - V(\phi)$ $\Rightarrow \rho + P = \dot\phi^2$

LHS=RHS

$$\Rightarrow -8\pi G H\left(\frac{d\phi^{(0)}}{dt}\right)^2 = \frac{8\pi G}{3}\left(\cancel{\frac{d\phi^{(0)}}{dt}}\right)\left(\frac{d^2\phi^{(0)}}{dt^2}\right) + V'\cancel{\frac{d\phi^{(0)}}{dt}}$$

$$\Rightarrow \boxed{\frac{d^2\phi^{(0)}}{dt^2} + 3H\frac{d\phi^{(0)}}{dt} + V' = 0}$$

Evolution of homogeneous scalar field in an expanding Universe

More useful in terms of conformal time

$d\eta = \frac{dt}{a} \Rightarrow \frac{d}{d\eta} = a\frac{d}{dt} \Rightarrow \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta}$ $\qquad \frac{d}{dt} = \frac{\dot{}}{a}$ $\qquad \frac{df}{dt} = \frac{\dot f}{a}$

$H = \frac{1}{a}\frac{da}{dt} = \frac{1}{a^2}\frac{da}{d\eta} = \frac{\dot a}{a^2}$

$$\frac{d^2}{dt^2} = \frac{1}{a}\frac{d}{d\eta}\left(\frac{1}{a}\frac{d}{d\eta}\right) = \frac{1}{a^2}\frac{d^2}{d\eta^2} - \frac{1}{a}\frac{\dot a}{a^2}\frac{d}{d\eta} = \frac{1}{a^2}\frac{d^2}{d\eta^2} - \frac{\dot a}{a^3}\frac{d}{d\eta} = \frac{1}{a^2}\frac{d^2}{d\eta^2} - \frac{H}{a}\frac{d}{d\eta}$$

$\cancel{\frac{d^2}{dt^2} = \frac{1}{a^2}}$ Symbolically: $\frac{d^2}{dt^2} = \frac{\ddot{}}{a^2} - \frac{H\dot{}}{a}$ $\qquad \cancel{\frac{d^2}{dt^2}}$ $\frac{d^2 f}{dt^2} = \frac{\ddot f}{a^2} - \frac{H\dot f}{a}$

$$\Rightarrow \frac{d^2\phi^{(0)}}{dt^2} + 3H\frac{d\phi^{(0)}}{dt} + V' = \frac{\ddot\phi^{(0)}}{a^2} - \frac{H\dot\phi^{(0)}}{a} + \frac{3H\dot\phi^{(0)}}{a} + V' = 0$$

$$\Rightarrow \boxed{\ddot{\phi}^{(0)} + 2aH\dot{\phi}^{(0)} + a^2 V' = 0}$$

Most inflationary models are slow-roll models

↳ $\phi^{(0)}$ and therefore H vary slowly

$$\eta \equiv \int_{t_e}^{t} \frac{dt'}{a(t')} = \int_{a_e}^{a} \frac{dt'}{da'} \frac{da'}{a'(t')} = \int_{a_e}^{a} da' \left(\frac{1}{a'}\frac{da'}{dt'}\right)^{-1} \frac{1}{a'} = \int_{a_e}^{a} \frac{da'}{a'^2 H} =$$

negative! $t < t_e$

$$= \int_{a_e}^{a} da \frac{1}{Ha^2} \simeq \frac{1}{H}\int_{a_e}^{a} \frac{da}{a^2} = \frac{1}{H}\left[-\frac{1}{a}\right]\Bigg|_{a_e}^{a} = \frac{1}{H}\left(-\frac{1}{a} + \frac{1}{a_e}\right) \simeq -\frac{1}{aH}$$

abuse of notation

$H \simeq \text{const}$

$a_e >> a$

$$\eta \simeq -\frac{1}{aH}$$

Two variables usually adopted to quantify slow-roll

$$\varepsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right) = \cancel{\frac{1}{H}} - \frac{1}{H^2}\frac{dH}{dt} = -\frac{1}{H^2}\frac{1}{a}\frac{dH}{d\eta} = -\frac{\dot{H}}{aH^2}$$

$\left[RD \quad a \propto \eta \quad H = \frac{1}{2}\right]$

$\varepsilon \longrightarrow 0$ as $\phi, H \longrightarrow$ const

H always decreasing $\longrightarrow \varepsilon > 0$ $\qquad$ $\varepsilon \ll 1$ during inflation ($\varepsilon < 1$ definition of inflation)

$$\delta \equiv \frac{1}{H}\frac{d^2\phi^{(0)}/dt^2}{d\phi^{(0)}/dt} = \frac{1}{H\left(\frac{1}{a}\dot{\phi}^{(0)}\right)}\left[\frac{\ddot{\phi}^{(0)}}{a^2} - \frac{H}{a}\dot{\phi}^{(0)}\right] = -\frac{a}{H\dot{\phi}^{(0)}}\left[\frac{H}{a}\dot{\phi}^{(0)} - \frac{\ddot{\phi}^{(0)}}{a^2}\right] =$$

$$= -\frac{1}{aH\dot{\phi}^{(0)}}\left[aH\dot{\phi}^{(0)} - \ddot{\phi}^{(0)}\right] = -\frac{1}{aH\dot{\phi}^{(0)}}\left[3aH\dot{\phi}^{(0)} + a^2V'\right]$$

$-\ddot{\phi}^{(0)} = 2aH\dot{\phi}^{(0)} + a^2V'$

$\delta \ll 1$ during inflation

End of inflation is typically defined as moment when $\varepsilon \simeq 1$

Inflationary gravitational waves

- Inflation solves horizon problem and correlates otherwise disconnected scales, and when these were causally connected it also generates (scalar and tensor) perturbations

scalar → couple to matter density radiation → inhomogeneities and anisotropies

tensor → fluctuations to the gravitational metric → GWs! (not coupled to density)

- Start by looking at tensor perturbations as they are much simpler (scalar fluctuations couple to energy density fluctuations)

Idea: QM fluctuations during inflation generate the scalar and tensor perturbations we can observe today

$$\phi^{(0)} + \delta\phi \qquad g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$$

$$\langle \delta\phi \rangle = 0$$

non-zero variance! → $$\text{“}\langle \delta\phi^2 \rangle - \langle \delta\phi \rangle^2\text{”} \neq 0 \qquad \text{“}\langle \delta g_{\mu\nu}^2 \rangle - \langle \delta g_{\mu\nu} \rangle^2\text{”} \neq 0$$

↳ use to set initial conditions generated by inflation

Quantization of simple harmonic oscillator

Let's start from a simpler example (we will try to write more complicated examples in this form)

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \xrightarrow[\text{quantization}]{} \hat{x} = v(\omega,t)\,\hat{a} + v^*(\omega,t)\,\hat{a}^\dagger$$

$x \longrightarrow \hat{x}$ quantum operator

v solves $\frac{d^2x}{dt^2} + \omega^2 x = 0$

so $v \propto e^{-i\omega t}$

acts on states of system → $\hat{a}$ quantum operator

$\hat{a}|0\rangle = 0$

vacuum state

$[\hat{a}, \hat{a}^\dagger] \equiv \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \qquad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$

$[\hat{x}, \hat{p}] = i$ where $p = \frac{dx}{dt}$ (if v normalized as $v(\omega, t) = \frac{e^{-i\omega t}}{\sqrt{2\omega}}$)

Look at quantum fluctuations of operator $\hat{x}$ in vacuum

$$\langle|\hat{x}|^2\rangle \equiv \langle 0|\hat{x}^\dagger x|0\rangle = \langle 0|(v^*\hat{a}^\dagger + \bar{v}\hat{a})(v\hat{a} + v^*\hat{a}^\dagger)|0\rangle =$$

$$= \cancel{\langle 0|v^*\hat{a}^\dagger \hat{a}|0\rangle}_{\hat{a}|0\rangle=0} + \cancel{\langle 0|v^*\hat{a}^\dagger v^*\hat{a}^\dagger|0\rangle}_{\langle 0|\hat{a}^\dagger = (\hat{a}|0\rangle)^\dagger = 0} + \cancel{\langle 0|v\hat{a}v\hat{a}|0\rangle}_{\hat{a}|0\rangle=0}$$

$\hat{a}\hat{a}^\dagger = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a} = [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger\hat{a}$

$$+ \langle 0|v\hat{a}\, v^*\hat{a}^\dagger|0\rangle = |v(\omega,t)|^2 \langle 0|\hat{a}\hat{a}^\dagger|0\rangle =$$

$$= |v(\omega,t)|^2 \langle 0|[\hat{a}, \hat{a}^\dagger] + \cancel{\hat{a}^\dagger\hat{a}}_{\hat{a}|0\rangle=0}|0\rangle = |v(\omega,t)|^2 \langle 0|1|0\rangle = |v(\omega,t)|^2$$

$$\Longrightarrow \langle|\hat{x}|^2\rangle = |v(\omega,t)|^2 = \left|\frac{e^{-i\omega t}}{\sqrt{2\omega}}\right|^2 = \frac{e^{-i\omega t}}{\sqrt{2\omega}}\frac{e^{i\omega t}}{\sqrt{2\omega}} = \frac{1}{2\omega}$$

non-zero variance!

Similar calculation for tensor perturbations

Tensor perturbations

Recall evolution equations for tensor perturbations

$$\begin{cases} \ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + K^2 h_+ = 0 \\ \ddot{h}_\times + 2\frac{\dot{a}}{a}\dot{h}_\times + K^2 h_\times = 0 \end{cases}$$

$\to$ not exactly simple harmonic oscillator, but we want to bring it into that form

Redefinition $\tilde{h} \equiv \frac{a h}{\sqrt{16\pi G}}$ $\qquad h = h_+, h_\times$

Take derivatives with respect to η

$$\frac{\dot{h}}{\sqrt{16\pi G}} = \frac{d}{d\eta}\left(\frac{\tilde{h}}{a}\right) = \frac{\dot{\tilde{h}}}{a} - \frac{\dot{a}}{a^2}\tilde{h} \qquad \left[\text{note: } \frac{\dot{a}}{a^2}\tilde{h} = \frac{1}{a}\frac{\dot{a}}{a}h = \frac{\mathcal{H}}{a}h = Hh\right]$$

$$\frac{\ddot{h}}{\sqrt{16\pi G}} = \frac{d}{d\eta}\left[\frac{\dot{\tilde{h}}}{a} - \frac{\dot{a}}{a^2}\tilde{h}\right] = \frac{\ddot{\tilde{h}}}{a} - \frac{\dot{a}}{a^2}\dot{\tilde{h}} - \tilde{h}\frac{d}{d\eta}\left(\frac{\dot{a}}{a^2}\right) - \frac{\dot{a}}{a^2}\dot{\tilde{h}} =$$

$$\frac{d}{d\eta}\left(\frac{\dot{a}}{a^2}\right) = \frac{\ddot{a}}{a^2} - \frac{2(\dot{a})^2}{a^3}$$

$$= \frac{\ddot{\tilde{h}}}{a} - \frac{\dot{a}}{a^2}\dot{\tilde{h}} - \frac{\ddot{a}}{a^2}\tilde{h} + \frac{2(\dot{a})^2}{a^3}\tilde{h} - \frac{\dot{a}}{a^2}\dot{\tilde{h}} =$$

$$= \frac{\ddot{\tilde{h}}}{a} - 2\frac{\dot{a}}{a^2}\dot{\tilde{h}} - \frac{\ddot{a}}{a^2}\tilde{h} + \frac{2(\dot{a})^2}{a^3}\tilde{h} = \frac{\ddot{\tilde{h}}}{a} - 2\frac{\dot{a}}{a^2}\dot{\tilde{h}} + \left[\frac{2(\dot{a})^2}{a^3} - \frac{\ddot{a}}{a^2}\right]\tilde{h} =$$

$$= \frac{\ddot{\tilde{h}}}{a} - 2H\dot{\tilde{h}} + \left[\frac{2(\dot{a})^2}{a^3} - \frac{\ddot{a}}{a^2}\right]\tilde{h}$$

Let's plug these into the equation for h

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} + K^2 h = 0 \implies \frac{\ddot{h}}{\sqrt{16\pi G}} + 2\frac{\dot{a}}{a}\frac{\dot{h}}{\sqrt{16\pi G}} + K^2\frac{h}{\sqrt{16\pi G}} = 0 \qquad = \frac{\tilde{h}}{a}$$

$$\implies \frac{\ddot{\tilde{h}}}{a} - 2\frac{\dot{a}}{a^2}\dot{\tilde{h}} - \frac{\ddot{a}}{a^2}\tilde{h} + \frac{2(\dot{a})^2}{a^3}\tilde{h} + 2\frac{\dot{a}}{a}\frac{\dot{\tilde{h}}}{a} - 2\frac{\dot{a}}{a}\frac{\dot{a}}{a^2}\tilde{h} + K^2\frac{\tilde{h}}{a} =$$

$$= \frac{\ddot{\tilde{h}}}{a} - \cancel{2\frac{\dot{a}}{a^2}\dot{\tilde{h}}} - \frac{\ddot{a}}{a^2}\tilde{h} + \cancel{\frac{2(\dot{a})^2}{a^3}\tilde{h}} + \cancel{2\frac{\dot{a}}{a^2}\dot{\tilde{h}}} - \cancel{\frac{2(\dot{a})^2}{a^3}\tilde{h}} + K^2\frac{\tilde{h}}{a} =$$

$$= \frac{\ddot{\tilde{h}}}{a} + K^2\frac{\tilde{h}}{a} - \frac{\ddot{a}}{a^2}\tilde{h} = \boxed{\frac{1}{a}\left[\ddot{\tilde{h}} + \left(K^2 - \frac{\ddot{a}}{a}\right)\tilde{h}\right] = 0}$$

It is of the simple harmonic oscillator form

Recall $\quad \frac{d^2x}{dt^2} + \omega^2 x = \ddot{x} + \omega^2 x = 0 \qquad$ (no damping term $\propto \dot{\tilde{h}}$)

So we can immediately write: $\boxed{\hat{\tilde{h}}(\bar{K},\eta) = v(K,\eta)\hat{a}_{\bar{K}} + v^*(K,\eta)\hat{a}^{+}_{\bar{K}}}$

where v solves $\boxed{\ddot{v} + \left(K^2 - \frac{\ddot{a}}{a}\right)v = 0}$

Since for the simple harmonic oscillator

$$\langle |\hat{x}|^2 \rangle = |v(\omega,t)|^2$$

We can write

equal-time correlator

$$\langle |\tilde{h}|^2 \rangle \equiv \langle \hat{\tilde{h}}^{+}(\bar{k},\eta)\, \hat{\tilde{h}}(\bar{k}',\eta) \rangle = |v(\bar{k},\eta)|^2 (2\pi)^3 \delta^3(\bar{k}-\bar{k}')$$

The main difference compared to the simple harmonic oscillator is the factor $(2\pi)^3\delta^3(\bar{k}-\bar{k}')$ because a quantum field can be thought of as a ~~collectio~~ collection of infinite simple harmonic oscillators, one for each $\bar{k}$, which are independent (uncorrelated) if the equations are linear

Since $\tilde{h} = \dfrac{ah}{\sqrt{16\pi G}} \longrightarrow h = \dfrac{\sqrt{16\pi G}}{a}\tilde{h}$

$$\langle \hat{h}^{+}(\bar{k},\eta)\, \hat{h}(\bar{k}',\eta) \rangle = \frac{16\pi G}{a^2} |v(k,\eta)|^2 (2\pi)^3 \delta^3(\bar{k}-\bar{k}') \equiv (2\pi)^3 P_h(k)\, \delta^3(\bar{k}-\bar{k}')$$

$$P_h(k) \equiv \frac{16\pi G}{a^2} |v(k,\eta)|^2$$ POWER SPECTRUM

(dimensionless power spectrum $\Delta(k) \equiv \dfrac{k^3 P(k)}{2\pi^2}$)

$\rightarrow$ quantifies variance/strength of fluctuations on a certain scale!

So now we just need to solve for $v(k,\eta)$ during inflation

$$\ddot{v} + \left(k^2 - \frac{\ddot{a}}{a}\right) v = 0$$

During inflation $\eta \simeq -\dfrac{1}{aH}$ $\left(= \displaystyle\int_{a_e}^{a} \frac{da}{Ha^2} \Big|_{a_e \gg a,\, H \simeq \text{const}} \right)$

Recall $d\eta = \dfrac{dt}{a} \rightarrow \dfrac{d}{dt} = \dfrac{1}{a}\dfrac{d}{d\eta} \rightarrow H = \dfrac{1}{a}\dfrac{\dot{a}}{a} = \dfrac{\dot{a}}{a^2}$ ~~$\dot{a} = a^2 H$~~

and also $H \simeq -\dfrac{1}{a\eta}$ $\Big\}$ $\dfrac{\dot{a}}{a^2} = -\dfrac{1}{a\eta} \Rightarrow \boxed{\dot{a} = -\dfrac{a}{\eta}}$

(only during inflation!)

$\Rightarrow$ ~~$\dfrac{\ddot{a}}{a} = \dfrac{1}{a}\dfrac{d}{d\eta}(\dot{a}) = \dfrac{1}{a}\dfrac{d}{d\eta}\left(-\dfrac{a}{\eta}\right) = \dfrac{1}{a}\left(-\dfrac{\dot{a}}{\eta^2} + \dfrac{a}{\eta^2}\right)$~~

$$\Rightarrow \frac{\ddot a}{a} = \frac{1}{a}\frac{d}{d\eta}(\dot a) \underset{\uparrow \dot a \simeq -\frac{a}{\eta}}{\simeq} \frac{1}{a}\frac{d}{d\eta}\left(-\frac{a}{\eta}\right) = -\frac{1}{a}\frac{d}{d\eta}\left(\frac{a}{\eta}\right) = -\frac{1}{a}\left(\frac{\dot a}{\eta} - \frac{a}{\eta^2}\right) \underset{\uparrow \dot a \simeq -\frac{a}{\eta}}{\simeq} -\frac{1}{a}\left(-\frac{a}{\eta^2} - \frac{a}{\eta^2}\right)$$

$$= -\frac{1}{a}\left(-\frac{2a}{\eta^2}\right) = \frac{2}{\eta^2}$$

$$\frac{\ddot a}{a} \simeq \frac{2}{\eta^2} \longrightarrow \ddot v + \left(k^2 - \frac{\ddot a}{a}\right)v = 0 \Longrightarrow \boxed{\ddot v + \left(k^2 - \frac{2}{\eta^2}\right)v = 0}$$

Initial conditions for $v(k,\eta)$: very early times before inflation has done most of its work

Recall $\eta = \int_{t_e}^{t} \frac{dt'}{a(t')}$ $\quad \eta_{prim} = \int_{t_e}^{t_*} \frac{dt'}{a(t')} < 0 \quad \eta_{tot} = \eta + \eta_{prim}$

At such early times $-\eta$ large, $-\eta \simeq \eta_{prim}$

$$\Rightarrow k^2 - \frac{2}{\eta^2} \simeq k^2 \longrightarrow \text{simple harmonic oscillator equation}$$

$\rightarrow$ properly normalized solution $v \propto \frac{e^{-ik\eta}}{\sqrt{2k}}$

Then with this boundary condition we can solve the full equation for v (exercise! Use $\tilde v = \frac{v}{\eta}$)

$$v(k,\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left[1 - \frac{i}{k\eta}\right] \xrightarrow[\substack{\text{sub-horizon}\\ \text{before inflation}\\ \text{has operated}\\ k|\eta| \gg 1}]{} \frac{e^{-ik\eta}}{\sqrt{2k}}$$

Let's check that this v is indeed solution!

Multiply by $\sqrt{2k}$, check $\frac{d^2}{d\eta^2}\left[e^{-ik\eta}\left(1 - \frac{i}{k\eta}\right)\right] + \left(k^2 - \frac{2}{\eta^2}\right)e^{-ik\eta}\left(1 - \frac{i}{k\eta}\right) \overset{?}{=} 0$

$$\frac{d}{d\eta}\left[e^{-ik\eta}\left(1 - \frac{i}{k\eta}\right)\right] = \frac{d}{d\eta}\left[e^{-ik\eta} - \frac{i}{k\eta}e^{-ik\eta}\right] = -ike^{-ik\eta} - \frac{ke^{-ik\eta}}{k\eta} + \frac{i}{k\eta^2}e^{-ik\eta} =$$

$$= -ike^{-ik\eta} - \frac{e^{-ik\eta}}{\eta} + \frac{i}{k\eta^2} e^{-ik\eta}$$

$$\frac{d^2}{d\eta^2}\left[e^{-ik\eta}\left(1-\frac{i}{k\eta}\right)\right] = \frac{d}{d\eta}\left[-ike^{-ik\eta} - \frac{e^{-ik\eta}}{\eta} + \frac{i}{k\eta^2} e^{-ik\eta}\right] =$$

$$= -k^2 e^{-ik\eta} + \frac{ik}{\eta} e^{-ik\eta} + \frac{e^{-ik\eta}}{\eta^2} + \frac{\cancel{k}}{\cancel{k}\eta^2} e^{-ik\eta} - \frac{2i}{k\eta^3} e^{-ik\eta} =$$

$$= -k^2 e^{-ik\eta} + \frac{ik}{\eta} e^{-ik\eta} + \frac{e^{-ik\eta}}{\eta^2} + \frac{e^{-ik\eta}}{\eta^2} - \frac{2i}{k\eta^3} e^{-ik\eta}$$

$$\frac{d^2}{d\eta^2}\left[e^{-ik\eta}\left(1-\frac{i}{k\eta}\right)\right] + \left(k^2 - \frac{2}{\eta^2}\right) e^{-ik\eta}\left(1-\frac{i}{k\eta}\right) =$$

$$= \cancel{-k^2 e^{-ik\eta}} + \cancel{\frac{ik}{\eta} e^{-ik\eta}} + \cancel{\frac{e^{-ik\eta}}{\eta^2}} + \cancel{\frac{e^{-ik\eta}}{\eta^2}} - \cancel{\frac{2i}{k\eta^3} e^{-ik\eta}} + \cancel{k^2 e^{-ik\eta}} - \cancel{\frac{ik}{\eta} e^{-ik\eta}}$$

$$\cancel{-\frac{2}{\eta^2} e^{-ik\eta}} + \cancel{\frac{2i}{k\eta^3} e^{-ik\eta}} = 0!$$

So the solution to the Mukhanov-Sasaki equation is indeed:

$$v = \frac{e^{-ik\eta}}{\sqrt{2k}}\left[1 - \frac{i}{k\eta}\right]$$

After inflation has done its job $k|\eta| \ll 1$

↳ mode has exited the horizon

Variance of the super-horizon GW amplitude: $\lim_{-k\eta\to 0} |v(k,\eta)|^2 \frac{16\pi G}{a^2}$

$$\lim_{-k\eta\to 0} v(k,\eta) = -\frac{e^{-ik\eta}}{\sqrt{2k}} \frac{i}{k\eta} = -\frac{ie^{-ik\eta}}{\sqrt{2k^3\eta^2}}$$

Recall then that $\hat{\tilde{h}} = v\hat{a}^{\dagger} + v^{*}\hat{a}$ $\quad \tilde{h} \equiv \frac{ah}{\sqrt{16\pi G}}$

$\Rightarrow h \propto \frac{v}{a}$ How does h evolve?

- Early times: (sub-horizon) $v \propto \frac{e^{-ik\eta}}{\sqrt{2k}}$ $h \propto \frac{v}{a} \propto \frac{1}{a}$ inflation reduces amplitude of mode

Late times: (after horizon exit) $v \propto \frac{e^{-ik\eta}}{k^{3/2}\eta}$ $h \propto \frac{v}{a} \propto \frac{1}{\eta a} \propto$ Const since $\eta \simeq -\frac{1}{aH}$

So after the mode exits the horizon it is roughly constant and so is the power spectrum (constant in time, not in k!)

this constant then sets initial conditions for GWs generated by inflation

$$P_h(k) = \frac{16\pi G}{a^2} \lim_{-k\eta \to 0} |v(k,\eta)|^2 =$$

$$= \frac{16\pi G}{a^2} \frac{1}{2k^3\eta^2} = \frac{8\pi G}{k^3 a^2 \eta^2} \simeq \frac{8\pi G H^2}{k^3}$$

$\eta \simeq -\frac{1}{aH}$, $a\eta \simeq -\frac{1}{H}$, $\frac{1}{a^2\eta^2} \simeq H^2$

$$\boxed{P_h(k) \simeq \frac{8\pi G H^2}{k^3}\bigg|_{\text{horizon exit : } k = aH}}$$

Scale-invariant spectrum since dimensionless power spectrum $\Delta \propto k^3 P(k) \propto$ Const

Detection of inflationary GWs would measure $H^2 \propto \rho \propto V$ during inflation

Typically $V \gtrsim 10^{15}$ GeV, well beyond what can be probed terrestrially

Note $H^2 \propto \frac{\rho}{M_{Pl}^2} \rightarrow P(k) \propto \frac{\rho}{M_{Pl}^4} \rightarrow$ Huge suppression!

Final remarks:

- Fluctuations in h are Gaussian as for simple harmonic oscillator
- $P_{h_+} = P_{h_\times} \rightarrow$ total power spectrum $= 2 \times P_h(k)$!