II Inflationary scalar perturbations

Previously we found the power spectrum of h emerging from inflation
Now we want to find the power spectrum of $\Psi = -\Phi \Rightarrow P_\Psi = P_\Phi$
Then using the relations found earlier we can relate this to the spectra of other variables

$$\Phi(k,\eta_i) = 2\Theta_0(k,\eta_i) = 2N_0(k,\eta_i)$$

$$\delta(k,\eta_i) = \delta_b(k,\eta_i) = 3\Theta_0(k,\eta_i)$$

$$\Theta_1(k,\eta_i) = N_1(k,\eta_i) = \frac{i v_b(k,\eta_i)}{3} = \frac{i v(k,\eta_i)}{3} = -\frac{k\,\Phi(k,\eta_i)}{6aH}$$

This is <u>difficult</u> because perturbations to the inflaton ϕ are coupled to Ψ. So we will start by ignoring this coupling

Strategies:

a) perturbations to ϕ ignoring Ψ
(Ψ negligible until ~~ϕ mode~~ mode moves outside the horizon)

$\downarrow$

Linear combination of $\delta\phi$ and Ψ conserved on super-horizon scales

$\downarrow$

Convert $\delta\phi$ spectrum to Ψ spectrum

b) Work in spatially flat slicing: gauge where g_{ij} is unperturbed

$\downarrow$

find gauge-invariant variable proportional to $\delta\phi$ in spatially flat slicing

$\downarrow$

Convert back to conformal Newtonian gauge

Scalar field perturbations around a smooth background

Start by looking at spectrum for $\delta\phi$ neglecting Ψ

$$\phi(\bar{x},t) = \underbrace{\phi^{(0)}(t)}_{\substack{\text{zero-order}\\\text{homogeneous}}} + \underbrace{\delta\phi(\bar{x},t)}_{\substack{\text{first-order}\\\text{perturbation}}}$$

Unperturbed metric $\quad g_{00} = -1 \quad g_{ij} = \delta_{ij} a^2 \quad$ (no Ψ, Φ)

Conservation of energy-momentum tensor

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu{}_{\alpha\mu} T^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu} T^\mu{}_\alpha = 0$$

$\hookrightarrow$ ${}^0{}_0$ component expanded to 1st order: equation for $\delta\phi$

Recall $\quad \Gamma^0{}_{ij} = \delta_{ij}\dot{a}a = \delta_{ij} a^2 H \qquad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \delta_{ij}\frac{\dot{a}}{a} = \delta_{ij} H$

All other Γs $= 0$

All Γs are 0 or 0th order, so ~~perturbations~~ 1st order pieces only come from perturbations to $T^\mu{}_\nu$ $\quad [\delta T^\mu{}_\nu]$

$\nu=0$ component of perturbed conservation equation

$$\nabla_\mu \delta T^\mu{}_0 = \partial_\mu \delta T^\mu{}_0 + \Gamma^\mu{}_{\alpha\mu}\delta T^\alpha{}_0 - \Gamma^\alpha{}_{0\mu}\delta T^\mu{}_\alpha =$$

$$\cancel{= \partial_0 \delta T^0{}_0 + \partial_i \delta T^i{}_0 + \Gamma^i{}_{0i}\delta T^0{}_0 - \ldots}$$

$$= \frac{\partial \delta T^0{}_0}{\partial t} + \underbrace{i k_i \delta T^i{}_0}_{\text{Fourier space}} + \underbrace{3H}_{\delta_{ii}=3} \delta T^0{}_0 - H\delta T^i{}_i$$

$$\Rightarrow \frac{\partial \delta T^0{}_0}{\partial t} + i k_i \delta T^i{}_0 + 3H\delta T^0{}_0 - H\delta T^i{}_i = 0$$

So we need the perturbed pieces of $T^\mu{}_\nu$

$$T^\alpha{}_\beta = g^{\alpha\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^\beta} - g^\alpha{}_\beta\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} + V(\phi)\right]$$

$$\delta T^i{}_0 = g^{i\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^0} - \cancelto{0}{g^i{}_0}\left[\dots\right] = g^{i\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^0} =$$

$$\underset{\text{0th order}}{=0} \quad \text{since } \nu=i \ (g^{i\nu}=a^{-2}\delta_{i\nu}) \text{ and } \frac{\partial\phi^{(0)}}{\partial x^i}=0 \Rightarrow T^{(0)i}{}_0 = 0$$

let's look at 1st order piece : set $\dfrac{\partial\phi}{\partial x^i} = \dfrac{\partial\delta\phi}{\partial x^i} = ik_i\delta\phi$

$$\cancel{\delta T^i{}_0 = g^{i\nu}\frac{\partial\delta\phi}{\partial x^\nu}\frac{\partial\delta\phi}{\partial x^0} = g^{ii}\frac{\partial\delta\phi}{\partial x^i}\frac{\partial\delta\phi}{\partial x^0} = a^{-2}ik_i\delta\phi\frac{\partial\delta\phi}{\partial t} = ik_i\frac{\partial\delta\phi^{(0)}}{\partial t}}$$

$\frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta} = \frac{\dot{}}{a}$

since $\dfrac{\partial\phi^{(0)}}{\partial x^i} = 0$ must set $\dfrac{\partial\phi}{\partial x^i} = \dfrac{\partial\delta\phi}{\partial x^i}$

$$\delta T^i{}_0 = \left[g^{i\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^0}\right]\Bigg|_{\text{1st order}} \simeq g^{i\nu}\frac{\partial\delta\phi}{\partial x^\nu}\frac{\partial\phi^{(0)}}{\partial x^0} = g^{ii}\frac{\partial\delta\phi}{\partial x^i}\frac{\partial\phi^{(0)}}{\partial t} =$$

$$\cancel{= a^{-2}ik_i\delta\phi} = a^{-2}\underbrace{ik_i}_{\text{Fourier space}}\delta\phi\frac{\partial\phi^{(0)}}{\partial t} = \frac{ik_i}{a^3}\frac{\partial\phi^{(0)}}{\partial\eta}\delta\phi$$

$\frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta} = \frac{\dot{}}{a}$

$$\delta T^i{}_0 = \frac{ik_i}{a^3}\frac{\partial\phi^{(0)}}{\partial\eta}\delta\phi = \frac{ik_i}{a^3}\dot{\phi}^{(0)}\delta\phi$$

$$T^0{}_0 = g^{00}\left(\frac{\partial\phi}{\partial x^0}\right)^2 - \frac{1}{2}g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} - V(\phi) = \frac{1}{2}g^{00}\left(\frac{\partial\phi}{\partial x^0}\right)^2 - \frac{1}{2}g^{ii}\left(\frac{\partial\phi}{\partial x^i}\right)^2 - V(\phi) =$$

$\phi = \phi^{(0)} + \delta\phi$, $g^{00} = -1$, $g^{ii} = a^{-2}$

$$= -\frac{1}{2}\left(\frac{\partial\phi^{(0)}}{\partial t} + \frac{\partial\delta\phi}{\partial t}\right)^2 - \frac{1}{2a^2}\cancelto{\text{2nd order}}{\frac{\partial\delta\phi}{\partial x^i}\frac{\partial\delta\phi}{\partial x^i}} - V(\phi^{(0)} + \delta\phi) \simeq$$

$$\simeq -\frac{1}{2}\left[\underbrace{\left(\frac{\partial \phi^{(0)}}{\partial t}\right)^2}_{\text{0th order}} + 2\frac{\partial \phi^{(0)}}{\partial t}\frac{\partial \delta\phi}{\partial t} + \cancel{\overset{\text{2nd order}}{\left(\frac{\partial \delta\phi}{\partial t}\right)^2}}\right] - \underbrace{\underbrace{V(\phi^{(0)})}_{\text{0th order}} - \delta\phi \frac{\partial V(\phi)}{\partial \phi}}_{\text{Taylor expansion}}$$

$$\Rightarrow \delta T^{0}{}_{\phi}\Big|_{\text{1st order}} \simeq -\frac{\partial \phi^{(0)}}{\partial t}\frac{\partial \delta\phi}{\partial t} - V'\delta\phi \underset{\uparrow \; \frac{d}{dt} = \frac{d_\eta}{a}}{=} -\frac{\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi$$

$$\delta T^{0}{}_{\phi} = \frac{-\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi$$

Exercise: $\delta T^{i}{}_{j}$

Summary

$$\delta T^{i}{}_{0} \simeq \frac{ik_i}{a^3}\phi^{(0)}\delta\phi \qquad \delta T^{0}{}_{0} \simeq \frac{-\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi$$

$$\delta T^{i}{}_{j} = \delta_{ij}\left(\frac{\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi\right)$$

Back to conservation equation

$$\frac{\partial \delta T^{0}{}_{0}}{\partial t} + ik_i\delta T^{i}{}_{0} + 3H\delta T^{0}{}_{0} - H\delta T^{i}{}_{i} = \left(\frac{\partial}{\partial t} + 3H\right)\delta T^{0}{}_{0} + ik_i\delta T^{i}{}_{0} - H\delta T^{i}{}_{i} =$$

$$= \left(\frac{1}{a}\frac{\partial}{\partial\eta} + 3H\right)\delta T^{0}{}_{0} + ik_i\delta T^{i}{}_{0} - H\delta T^{i}{}_{i} = 0$$

Putting everything together

$$\left(\frac{1}{a}\frac{\partial}{\partial\eta} + 3H\right)\left(\frac{-\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi\right) - \frac{k^2}{a^3}\phi^{(0)}\delta\phi \underbrace{-3H}_{\delta_{ii}}\left(\frac{\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi\right) = 0$$

$$\hookrightarrow = \frac{1}{a}\frac{\partial}{\partial\eta}\left[\frac{-\dot\phi^{(0)}\dot{\delta\phi}}{a^2}\right] - \frac{1}{a}\frac{\partial}{\partial\eta}(V'\delta\phi) - 3H\frac{\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - \cancel{3HV'\delta\phi} - \frac{k^2}{a^3}\phi^{(0)}\delta\phi$$

$$-3H\frac{\dot\phi^{(0)}\dot{\delta\phi}}{a^2} + \cancel{3HV'\delta\phi}.$$

$$= \frac{1}{a}\frac{\partial}{\partial\eta}\left[\frac{-\dot\phi^{(0)}\delta\dot\phi}{a^2}\right] - \frac{1}{a}\frac{\partial}{\partial\eta}(V'\delta\phi) - \frac{6H\dot\phi^{(0)}\delta\dot\phi}{a^2} - \frac{k^2}{a^3}\dot\phi^{(0)}\delta\phi = *$$

$$\frac{1}{a}\frac{\partial}{\partial\eta}\left[\frac{-\dot\phi^{(0)}\delta\dot\phi}{a^2}\right] = -\frac{\ddot\phi^{(0)}\delta\dot\phi}{a^3} - \frac{\dot\phi^{(0)}\delta\ddot\phi}{a^3} + \frac{2\dot\phi^{(0)}\delta\dot\phi}{a^4}\dot a =$$

$$\frac{\dot a}{a^4} = \frac{1}{a^2}\frac{\dot a}{a^2} = \frac{H}{a^2}$$

since $H = \frac{1}{a}\frac{da}{dt} = \frac{\dot a}{a^2}$

$$= -\frac{\ddot\phi^{(0)}\delta\dot\phi}{a^3} - \frac{\dot\phi^{(0)}\delta\ddot\phi}{a^3} + \frac{2H\dot\phi^{(0)}\delta\dot\phi}{a^2}$$

$$-\frac{1}{a}\frac{\partial}{\partial\eta}(V'\delta\phi) = -\frac{1}{a}\left[\frac{\partial V'}{\partial\eta}\delta\phi + V'\delta\dot\phi\right] = -\frac{1}{a}\left[\underbrace{\frac{\partial V'}{\partial\phi^{(0)}}}_{V''}\underbrace{\frac{\partial\phi^{(0)}}{\partial\eta}}_{\dot\phi^{(0)}}\delta\phi + V'\delta\dot\phi\right] =$$

$$= -\frac{1}{a}\left(V''\dot\phi^{(0)}\delta\phi + V'\delta\dot\phi\right) = -\frac{1}{a}V''\dot\phi^{(0)}\delta\phi - \frac{1}{a}V'\delta\dot\phi$$

$$* = -\frac{\ddot\phi^{(0)}\delta\dot\phi}{a^3} - \frac{\dot\phi^{(0)}\delta\ddot\phi}{a^3} + \frac{2H\dot\phi^{(0)}\delta\dot\phi}{a^2} - \frac{V''\dot\phi^{(0)}\delta\phi}{a} - \frac{V'\delta\dot\phi}{a}$$

$$-\frac{6H\dot\phi^{(0)}\delta\dot\phi}{a^2} - \frac{k^2}{a^3}\dot\phi^{(0)}\delta\phi = 0$$

$\Rightarrow$ $\times a^3$

$$-\dot\phi^{(0)}\delta\ddot\phi + \delta\dot\phi\underbrace{\left(-\ddot\phi^{(0)} - 4aH\dot\phi^{(0)} - a^2V'\right)}_{\text{scalar field equation of motion}} + \delta\phi\left(\cancel{-a^2\underbrace{V''}_{\text{small, } \propto\epsilon,\delta}} - k^2\dot\phi^{(0)}\right) = 0$$

$$\ddot\phi^{(0)} + 2aH\dot\phi^{(0)} + a^2V' = 0 \Rightarrow -\ddot\phi^{(0)} - 4aH\dot\phi^{(0)} - a^2V' = -2aH\dot\phi$$

$$\Rightarrow \cancel{-\dot\phi^{(0)}}\delta\ddot\phi - 2aH\cancel{\dot\phi^{(0)}}\delta\dot\phi - k^2\cancel{\dot\phi^{(0)}}\delta\phi = 0$$

$$\Rightarrow \boxed{\delta\ddot\phi + 2aH\delta\dot\phi + k^2\delta\phi = 0}$$

Equation for perturbations to inflaton scalar field in unperturbed Universe

Now recall what was the evolution equation for tensor perturbations

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} + k^2 h = h'' + 2aHh' + k^2 h = 0$$

Identical in form to $\delta\phi$ equation! So for $\delta\phi$ we can borrow the results from tensor perturbations, minus the factor of $16\pi G$ which we don't need here

$$P_h(k) = \frac{16\pi G}{a^2}\frac{1}{2k^3\eta^2} \underset{\uparrow \; \eta \simeq -\frac{1}{aH}}{\simeq} \frac{8\pi G H^2}{k^3} \xRightarrow[\times \frac{1}{16\pi G}]{} \boxed{P_{\delta\phi}(k) \simeq \frac{H^2}{2k^3}}$$

So also the power spectrum of scalar field fluctuations from inflation is (nearly...) scale-invariant, $\Delta \propto k^3 P(k) \propto \text{const}$

Super-horizon perturbations

So far we neglected Ψ, which is valid when the perturbation is sub-horizon ($k \gg aH$), but when a mode exits the horizon Ψ becomes important and we can find a linear combination of $\delta\phi$ and Ψ which is conserved.

Symbolically, evolution of inflationary perturbations:

$$(\text{mostly})\ \delta\phi \longrightarrow \Psi \text{“}+\text{”} \delta\phi \ / \ \Psi \text{“}+\text{”} \delta T^\mu{}_\nu$$

Goal: find linear combination of Ψ and $\delta\phi$ conserved ~~at horizon~~ ~~crossing, then~~ on sub-horizon scales, will depend on $\delta\phi$ at horizon crossing, evaluate it after inflation in terms of $\underbrace{\Psi}_{\text{after inflation}}$ $\underbrace{(\propto \delta\phi)}_{\text{before inflation}}$, finally

relate P_Ψ to $P_{\delta\phi}$

$$\hookrightarrow = \frac{H^2}{2k^3}$$

Start again from covariant conservation of energy-momentum:

$$\underbrace{\frac{\partial \delta T^0{}_0}{\partial t} + ik_i \delta T^i{}_0 + 3H\delta T^0{}_0 - H\delta T^i{}_i}_{\text{saw earlier}} = -3(P+\rho)\frac{\partial \Psi}{\partial t}$$

Why were we justified in neglecting Ψ previously? If $(P+\rho)\frac{\partial\Psi}{\partial t} \ll$ RHS, this was fine. Example: 1st term

$$\frac{\partial \delta T^0{}_0}{\partial t} \gg (P+\rho)\frac{\partial \Psi}{\partial t} \implies \Psi \ll \frac{\delta T^0{}_0}{P+\rho}$$

Is this true? Look at perturbed Einstein equations we found

$$k^2\Phi + 3\frac{\dot a}{a}\left(\dot\Phi - \Psi\frac{\dot a}{a}\right) = -4\pi G a^2 \delta T^0{}_0 \xrightarrow[\frac{\dot a}{a}=aH]{\Phi=-\Psi} \boxed{k^2\Psi + 3aH(\dot\Psi + aH\Psi) = 4\pi G a^2 \delta T^0{}_0} *$$

$*$ LHS $\sim k^2\Psi \sim a^2H^2\Psi$ at horizon crossing

$$\Rightarrow \cancel{a^2}H^2\Psi \sim G\cancel{a^2}\delta T^0{}_0 \implies \Psi \sim \frac{G\delta T^0{}_0}{H^2} \sim \frac{\delta T^0{}_0}{\rho} = \qquad \left[H^2 \sim G\rho \Rightarrow \frac{G}{H^2} \sim \frac{1}{\rho}\right]$$

$$= \frac{P+\rho}{\rho}\left(\frac{\delta T^0{}_0}{P+\rho}\right)$$

For $\Psi \ll \frac{\delta T^0{}_0}{P+\rho}$ we require $\frac{P+\rho}{\rho} \ll 1$ which is the case during inflation since $P \simeq -\rho$ [in fact $\frac{P+\rho}{\rho} = \frac{2\varepsilon}{3} \ll 1$]

So for slow-roll models of inflation we can safely neglect Ψ when computing perturbations $\delta\phi$ FOR SUBHORIZON MODES

However at some point $\underbrace{\frac{P+\rho}{\rho}}_{=1+w} \ll 1$ won't hold anymore

$\to \sim 0$ during inflation $w \simeq -1$

$\to \frac{4}{3}$ during RD

E.g. radiation domination

$$\delta T^0{}_0 = -4\rho_r \Theta_0 \qquad \rho + p = \frac{4\rho_r}{3} \Rightarrow \frac{\delta T^0{}_0}{\rho+p} \sim -3\Theta_0$$

~~Ψ~~ But we know that

$$-\Phi(k, \eta_i) = \Psi(k, \eta_i) = -2\Theta_0(k, \eta_i)$$

$\hookrightarrow \eta_i$ early on but after inflation has operated!

Not true that $\Psi = -2\Theta_0 \ll \frac{\delta T^0{}_0}{\rho+p} \sim -3\Theta_0$!!!

Before the end of inflation Ψ has to grow in importance relative to $\delta T^0{}_0$: perturbations to the metric need to become comparable in importance to those in the energy-momentum tensor!

Define new variable

$$\zeta \equiv -\frac{ik_i \delta T^0{}_i H}{k^2(\rho+P)} - \Psi$$

For sub-horizon modes and modes which just crossed the horizon

$$\zeta_{sub} = \frac{-ik_i \delta T^0{}_i H}{k^2(\rho+P)} \underbrace{\cancel{-\Psi}}^{\text{negligible}} \underset{\Psi\to 0}{\simeq} \frac{-i\cancel{k_i}}{\cancel{k^2}} H \times -\frac{ik_i}{a^3} \dot{\phi}^{(0)} \delta\phi \left(\frac{a}{\dot{\phi}^{(0)}}\right)^2 = -\frac{aH\delta\phi}{\dot{\phi}^{(0)}}$$

$$P + \rho = \left(\frac{d\phi^{(0)}}{dt}\right)^2 = \frac{1}{a^2}\left(\dot{\phi}^{(0)}\right)^2$$

$$-\delta T^i{}_0 = +\delta T^0{}_i = -\frac{ik_i}{a^3} \dot{\phi}^{(0)} \delta\phi$$

After inflation ends (exercise!)

$$ik_i \delta T^0{}_i = 4ak\rho_r\Theta_1 \longrightarrow \delta T^0{}_i = -i\frac{4ak\rho_r\Theta_1}{k_i}$$

$$\zeta_{super} = \frac{-ik_i \delta T^0{}_i H}{k^2(\rho+P)} - \Psi \simeq -\frac{i\cancel{k_i}}{k^2} \times -\frac{i4ak\rho_r\Theta_1}{\cancel{k_i}} \frac{H}{(\rho_r + P_r)} - \Psi$$

$$= -\frac{4aH}{k}\Theta_1 \frac{\rho_r}{(\rho_r+P_r)} - \Psi \underset{\uparrow}{=} -\frac{3aH\Theta_1}{k} - \Psi \underset{\uparrow}{=} -\frac{3\cancel{aH}}{\cancel{k}}\frac{\cancel{k}\Psi}{6\cancel{aH}} - \Psi = -\frac{1}{2}\Psi - \Psi = -\frac{3}{2}\Psi$$

$$\frac{\rho_r + P_r}{\rho_r} = 1 + w_r = \frac{4}{3} \qquad \Theta_1(k, \eta_i) = -\frac{k\Phi(k,\eta_i)}{6aH} = \frac{k\Psi}{6aH}(k, \eta_i)$$

So ζ goes from $\simeq -\frac{aH\delta\phi}{\dot\phi^{(0)}}$ to $\simeq -\frac{3}{2}\Psi$

before and around horizon crossing

well after the end of inflation, on superhorizon scales

Why is ζ important? It is conserved for super-horizon perturbations (show later)

Since it is conserved

$$-\frac{aH\delta\phi}{\dot\phi^{(0)}}\Big|_{\text{horizon crossing}} = -\frac{3}{2}\Psi\Big|_{\text{post inflation}}$$

$$\Rightarrow \Psi\Big|_{\text{post inflation}} = \frac{2aH}{3}\frac{\delta\phi}{\dot\phi^{(0)}}\Big|_{\text{horizon crossing}}$$

But we care about the variance of Ψ (power spectrum)

$$\Rightarrow P_\Psi = \frac{4}{9}\frac{a^2H^2}{\dot\phi^{(0)2}}P_{\delta\phi}\Big|_{\text{horizon crossing}}$$

where ~~has~~ a mode k crosses the horizon when $k = aH$

$$P_\Psi = \frac{4}{9}\left(\frac{aH}{\dot\phi^{(0)}}\right)^2 P_{\delta\phi}\Big|_{aH=k} \underset{P_{\delta\phi}=\frac{H^2}{2k^3}}{=} \frac{4}{9}\left(\frac{aH}{\dot\phi^{(0)}}\right)^2\frac{H^2}{2k^3}\Big|_{aH=k} = \frac{2}{9}\left(\frac{aH^2}{\dot\phi^{(0)}}\right)^2\frac{1}{k^3}$$

Exercise: $\left(\frac{aH}{\dot\phi^{(0)}}\right)^2 \simeq \frac{4\pi G}{\varepsilon}$

$$\Rightarrow P_\Psi(k) \underset{\Psi\simeq-\Phi}{\simeq} P_\Phi(k) = \frac{2}{9}\frac{4\pi G}{\varepsilon}\frac{H^2}{k^3} = \frac{8\pi G}{9k^3}\frac{H^2}{\varepsilon}\Big|_{aH=k}$$

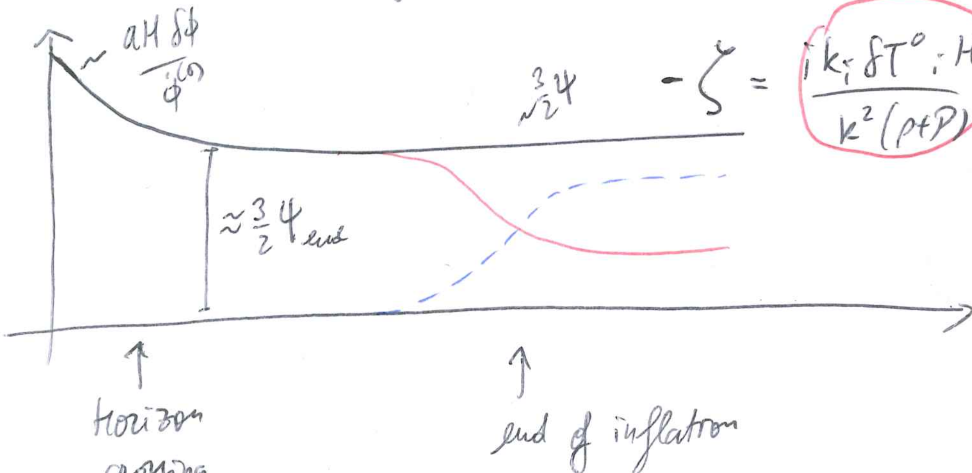
Recall $P_h \sim \frac{8\pi G H^2}{k^3} \longrightarrow \frac{P_{\Psi/\Phi}}{P_h} \sim \frac{1}{\varepsilon} \gg 1$

So we expect scalar modes to dominate over tensor modes in slow-roll inflation

Exercise: $\varepsilon \simeq \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$

$$\Rightarrow P_\psi(k) = P_\zeta(k) = \frac{8\pi G}{9k^3}\frac{H^2}{\varepsilon}\Big|_{aH=k} \simeq \frac{128\pi^2 G^2}{9k^3}\left(\frac{H^2 V^2}{V'^2}\right)\Big|_{aH=k}$$

Conservation of ζ for super-horizon modes, schematically



$$-\zeta = \frac{ik_i\,\delta T^0{}_i\,H}{k^2(\rho+P)} + \psi$$

Sub-horizon: QM fluctuations set up in $\phi^{(0)} \Rightarrow \delta\phi \Rightarrow \frac{ik_i\delta T^0{}_i H}{k^2(\rho+P)} \simeq \frac{aH\delta\phi}{\dot\phi^{(0)}}$

Negligible perturbations to the metric

Past horizon crossing: ζ conserved, but relative contribution of ψ to $\delta T^0{}_i$ increases, finally $\zeta \approx -\frac{3}{2}\psi$

Now we just need to show that ζ is indeed conserved on super-horizon scales. Go back to conservation equation (optional)

$\delta T^i{}_0 = ik_i\dot\phi^{(0)}\delta\phi/a^3 \sim k_i \Rightarrow ik_i\delta T^i{}_0 \sim k^2$ negligible on large scales

$$\frac{\partial \delta T^0{}_0}{\partial t} + \cancel{ik_i\,\delta T^i{}_0} + 3H\delta T^0{}_0 - H\delta T^i{}_i = -3(P+\rho)\frac{\partial\psi}{\partial t}$$

On large scales (exercise) $\frac{ik_i\,\delta T^0{}_i\,H}{k^2} \simeq \frac{\delta T^0{}_0}{3}$

$$\Rightarrow \zeta = \frac{ik_i\,\delta T^0{}_i\,H}{k^2(\rho+P)} - \psi \simeq -\psi - \frac{1}{3}\frac{\delta T^0{}_0}{\rho+P}$$

large scales

Combine the two

$$\begin{cases} \dfrac{\partial \delta T^0{}_0}{\partial t} + 3H\delta T^0{}_0 - H\delta T^i{}_i = -3(P+\rho)\dfrac{\partial \psi}{\partial t} \\ \zeta \simeq -\psi - \dfrac{1}{3}\dfrac{\delta T^0{}_0}{\rho+P} \quad \rightarrow \psi \simeq -\zeta - \dfrac{1}{3}\dfrac{\delta T^0{}_0}{\rho+P} \end{cases}$$

$$\hookrightarrow \cancel{\frac{\partial \delta T^0{}_0}{\partial t}} + 3H\delta T^0{}_0 - H\delta T^i{}_i = -3(P+\rho)\frac{\partial}{\partial t}\left[-\zeta - \frac{1}{3}\frac{\delta T^0{}_0}{\rho+P}\right] =$$

$$= 3(P+\rho)\frac{\partial}{\partial t}\left[\zeta + \frac{1}{3}\frac{\delta T^0{}_0}{\rho+P}\right] = 3(P+\rho)\frac{\partial \zeta}{\partial t} + (P+\rho)\frac{\partial}{\partial t}\left[\frac{\delta T^0{}_0}{P+\rho}\right] =$$

$$= 3(P+\rho)\frac{\partial \zeta}{\partial t} + \cancel{\frac{\partial \delta T^0{}_0}{\partial t}} - \delta T^0{}_0 \frac{\cancel{(P+\rho)}}{(P+\rho)^2}\left(\frac{d\rho}{dt} + \frac{dP}{dt}\right)$$

$$\Rightarrow \delta T^0{}_0\left[3H + \frac{1}{\rho+P}\left(\frac{d\rho}{dt} + \frac{dP}{dt}\right)\right] - H\delta T^i{}_i = 3(P+\rho)\frac{\partial \zeta}{\partial t}$$

Continuity equation $\qquad \dot\rho + 3H\rho(1+w) = 0 \Rightarrow \dfrac{d\rho}{dt} + 3H\rho\left(1 + \dfrac{P}{\rho}\right) = 0$

$$\Rightarrow \frac{d\rho}{dt} = -3H(\rho+P)$$

so $\rightarrow$ $\cancel{3H} +$ $3H + \dfrac{1}{\rho+P}\left(\dfrac{d\rho}{dt} + \dfrac{dP}{dt}\right) = \cancel{3H} \cancel{-3H} + \dfrac{1}{\rho+P}\dfrac{dP}{dt}$

$$\Rightarrow \frac{\delta T^0{}_0}{\rho+P}\frac{dP}{dt} - H\delta T^i{}_i = 3(P+\rho)\frac{\partial \zeta}{\partial t}$$

$$\Rightarrow \frac{\partial \zeta}{\partial t} = -\frac{1}{3(\rho+P)^2}\left[H(\rho+P)\delta T^i{}_i - \delta T^0{}_0\frac{dP}{dt}\right]$$

$H(\rho+P) = -\dfrac{1}{3}\dfrac{d\rho}{dt}$ from continuity equation

$$\Rightarrow [\ldots] \propto -\frac{1}{3}\frac{d\rho}{dt}\delta T^i{}_i - \delta T^0{}_0\frac{dP}{dt} \propto \frac{\delta T^i{}_i}{3} + \delta T^0{}_0\frac{dP}{d\rho} \propto \delta P - \frac{dP}{d\rho}\delta\rho$$

$\delta T^i{}_i = 3\delta P \qquad \delta T^0{}_0 = -\delta\rho$

so $\frac{\partial \zeta}{\partial t} \propto \delta P - \frac{dP}{d\rho}\delta\rho = 0$ for adiabatic perturbations!

~~Example~~: Recall for adiabatic perturbations the number density ratio between species is identical everywhere

so $\frac{\partial \zeta}{\partial t} = 0$ on super-horizon scales!

Spatially flat slicing

A cleaner way to obtain all the previous results is to use gauge transformations and gauge-invariant variables

In Newtonian gauge $\delta\phi$ and ψ are coupled

Spatially flat gauge

$$ds^2 = -(1+2A)\,dt^2 + 2a\frac{\partial B}{\partial x^i}dx^i dt + \delta_{ij}a^2 dx^i dx^j$$

In this gauge the equation we found previously are exact with no approximations required

$$\ddot{\delta\phi} + 2aH\,\dot{\delta\phi} + k^2\delta\phi = 0 \implies P_{\delta\phi} = \frac{H^2}{2k^3}$$

Bardeen identified two gauge-invariant variables

Bardeen's velocity (gauge-invariant!)

~~$v \equiv ikB - ik\frac{\dot\phi^{(0)}\delta\phi}{(\rho+P)a^2}$ in spatially flat slicing~~

$$v \equiv ikB + \frac{\hat{k}^i T^0{}_i}{(\rho+P)a} = ikB - ik\frac{\dot\phi^{(0)}\delta\phi}{(\rho+P)a^2}$$ in spatially flat slicing

$\delta T^0{}_i = \frac{ik_i}{a}\dot\phi\,\delta\phi$

Bardeen's potential

$$\Phi_H = -\psi + aH(B - \dot{E}) = aHB \qquad (\psi = 0,\ E = 0)$$

Any linear combination of Φ_H and v is gauge-invariant

Take: $$\zeta \equiv -\Phi_H - \frac{iaH}{k} v = -aHB - \frac{iaH}{k} ikB - \frac{aH\dot{\phi}^{(0)}\delta\phi}{(\rho+P)a^2} =$$

$$= \frac{-aH\dot{\phi}^{(0)}\delta\phi}{(\rho+P)a^2} \simeq \frac{-aH\,\delta\phi}{\dot{\phi}^{(0)}}$$

$\rho+P = \left(\frac{\dot{\phi}^{(0)}}{a}\right)^2$

$$\Rightarrow P_\zeta = \left(\frac{aH}{\dot{\phi}^{(0)}}\right)^2 P_{\delta\phi} = \frac{4\pi G}{\varepsilon}\frac{H^2}{2k^3} = \frac{2\pi G H^2}{\varepsilon k^3}\Big|_{aH=k}$$

power spectrum of a gauge-invariant quantity!!!

$P_{\delta\phi} = \frac{H^2}{2k^3}$, $\left(\frac{aH}{\dot{\phi}^{(0)}}\right)^2 = \frac{4\pi G}{\varepsilon}$

In conformal Newtonian gauge

$\Phi_H = -\Phi$ so $\zeta = -\Phi_H - \frac{iaH}{k} v = \frac{-ik_i\delta T^0{}_i H}{k^2(\rho+P)} - \Psi$ as we saw earlier

We also saw that after inflation $\zeta \simeq -\frac{3}{2}\Psi \simeq \frac{3}{2}\Phi$

$$\Rightarrow P_\Phi = \frac{4}{9} P_\zeta = \frac{8\pi G H^2}{9\varepsilon k^3}\Big|_{aH=k}$$ ✓ Matches previous result!

Physical interpretation of Φ_H: $\frac{4k^2\Phi_H}{a^2}$ is curvature of 3D space at fixed time $\rightarrow$ Perturbations to Φ_H are curvature perturbations (even though space-time flat to 0th order)

In a comoving gauge $v=0 \rightarrow \zeta = \Phi_H$ so ζ shares an interpretation as curvature: scalar perturbations from inflation often called curvature perturbations!

Spectral indices

So far

$\langle \Phi(\bar{k}) \rangle = 0$ $\quad\langle \Phi(\bar{k})\,\Phi^*(\bar{k}')\rangle = (2\pi)^3 P_\Phi(k)\,\delta^3(\bar{k}-\bar{k}')$

$$P_\Phi(k) = \frac{8\pi G H^2}{9\varepsilon k^3}\Big|_{aH=k} \qquad P_h(k) = \frac{8\pi G H^2}{k^3}$$

$P_\Phi \propto \frac{1}{\varepsilon}$ where $\varepsilon = -\frac{\dot{H}}{aH^2} \ll 1$ during inflation

quasi de-Sitter $\Downarrow$ quasi scale-invariant

- $k^3 P_\Phi(k) \approx$ const $\rightarrow$ scale invariant

However in reality there are small deviations from scale-invariance:
(because inflation is not perfectly de Sitter $\Rightarrow \eta = -\frac{1}{aH}$: inflation has to end!)

$$P_\Phi(k) = \frac{8\pi}{9k^3} \frac{H^2}{\varepsilon M_{Pl}^2}\Big|_{aH=k} \equiv \frac{50\pi^2}{9k^3}\left(\frac{k}{H_0}\right)^{n_s-1} \delta_H^2 \underbrace{\left(\frac{\Omega_m}{D_1(a=1)}\right)^2}_{\text{later, growth function}}$$

$$P_h(k) = \frac{8\pi H^2}{k^3 M_{Pl}^2}\Big|_{aH=k} \equiv A_T\, k^{n_T-3}$$

- δ_H, A_T scalar and tensor amplitudes

  n_s, n_T scalar and tensor tilts $\rightarrow$ scale-invariance $n_s=1$, $n_T=0$

n_s and n_T are related to the slow-roll parameters

Since $P_h(k) = A_T k^{n_T-3} \Rightarrow \frac{d\ln P_h}{d\ln k} = n_T - 3$ $\quad \left[\frac{k}{P_h}\frac{dP_h}{dk} = \frac{\cancel{k}}{\cancel{A_T k^{n_T-3}}}(n_T-3)\cancel{A_T k^{n_T-2}}\right]$

$\ln P_h(k) = \ln(A_T) + (n_T-3)\ln k$

- $$n_T - \cancel{3} = \frac{d\ln P_h}{d\ln k} = \frac{d}{d\ln k}\left[\ln\left(\frac{H^2}{2k^3}\right)\right] = \frac{d}{d\ln k}\left[2\ln H - 3\ln k\right] = -\cancel{3} + 2\frac{d\ln(H)}{d\ln(k)}$$

$$\Rightarrow n_T = 2\frac{d\ln(H)}{d\ln(k)}$$

$$\frac{d\ln(H)}{d\ln(k)}\Big|_{aH=k} = \frac{k}{H}\frac{dH}{dk} = \frac{k}{H}\frac{dH}{d\eta}\frac{d\eta}{dk}\Big|_{aH=k} = -\frac{k}{H}\frac{aH^2\varepsilon}{k^2}\Big|_{aH=k} = -\frac{aH}{k}\varepsilon\Big|_{aH=k} = -\varepsilon$$

$$\dot{H} \equiv -aH^2\varepsilon \rightarrow \frac{dH}{d\eta} = -aH^2\varepsilon$$

$$\frac{d\eta}{dk}\Big|_{aH=k} \underset{\eta \simeq -\frac{1}{aH}}{=} -\frac{d(aH)^{-1}}{dk}\Big|_{aH=k} = -\frac{d}{dk}\left(\frac{1}{k}\right) = \frac{1}{k^2}$$

- $$\Rightarrow \boxed{n_T = 2\frac{d\ln(H)}{d\ln(k)} = -2\varepsilon}$$

So expect very small tensor tilt $n_T \ll 1$

Similarly for n_s

$$n_s - 1 = \frac{d \ln P_\Phi}{d\ln k} = \frac{d}{d\ln k}\left[\ln(H^2) - \ln(\varepsilon)\right]$$

$$\frac{d\ln(H^2)}{d\ln k} = 2\frac{d\ln H}{d\ln k} = -2\varepsilon \qquad \frac{d\ln(\varepsilon)}{d\ln k} = -2(\varepsilon+\delta) \qquad \text{exercise!}$$

$$\Rightarrow n_s - 1 \simeq -2\varepsilon - 2\varepsilon + 2\delta \Rightarrow \boxed{n_s \simeq 1 - 4\varepsilon + 2\delta}$$

Note $\frac{P_\Phi}{P_h} \sim \frac{1}{\varepsilon}$ and $n_T \sim \varepsilon$: this is a robust prediction of single-field slow-roll inflation!

~~Worked~~ Tensor-to-scalar ratio : $r \equiv \frac{\Delta_h^2}{\Delta_\Phi^2} \simeq 16\varepsilon$

Summary: inflationary observables (see 0907.5424 arXiv)

$$\varepsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{\dot H}{aH^2} = -\frac{d\ln H}{dN} < 1 \qquad \text{where } dN = d\ln a = H dt$$

↳ number of e-folds

$$\delta \equiv \frac{1}{H}\frac{d^2\phi^{(0)}/dt^2}{d\phi^{(0)}/dt} = \varepsilon - \frac{1}{2\varepsilon}\frac{d\varepsilon}{d\eta} \qquad |\eta| < 1$$

de Sitter limit : $\varepsilon \to 0$ $\left[\frac{\ddot a}{a} = H^2(1-\varepsilon)\right]$

$\varepsilon \ll 1$: potential very flat $|\eta| \ll 1$: ~~change~~ fractional change of ε per e-fold small

Potential slow-roll parameters

$$\varepsilon_V(\phi) \equiv \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2$$

$$\eta_V(\phi) \equiv M_{Pl}^2 \frac{V''}{V}$$

slow-roll: $\varepsilon_V, |\eta_V| \ll 1$

In the slow-roll regime

$\epsilon \approx \epsilon_V \qquad \eta \approx \eta_V - \epsilon_V$ (Exercise) $\qquad H^2 \approx \frac{V(\phi)}{3} \approx$ const

$\dot{\phi} \approx -\frac{V'}{3H}$

$\hookrightarrow (|\ddot{\phi}| \ll |3H\dot{\phi}|, |V'|)$

Inflation ends when

$\epsilon(\phi_{end}) \equiv 1 \qquad \epsilon_V(\phi_{end}) \approx 1$

$$N(\phi) \equiv \ln \frac{a_{end}}{a} = \int_t^{t_{end}} dt\, H = \int_\phi^{\phi_{end}} \frac{dt}{d\phi} d\phi\, H = \int_\phi^{\phi_{end}} d\phi \frac{H}{\dot{\phi}} \approx \int_{\phi_{end}}^{\phi} d\phi \frac{V}{V'} =$$

↑ number of e-folds before inflation ends

$$= \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\epsilon}} \approx \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\epsilon_V}} \gtrsim 60$$

Fluctuations observed in the CMB exited horizon $\approx 40-60$ e-folds before the end of inflation

$$\int_{\phi_{end}}^{\phi_{cmb}} \frac{d\phi}{\sqrt{2\epsilon_V}} = N_{cmb} \approx 40-60$$

$$\boxed{\begin{aligned} & n_s - 1 \simeq 2\eta_V^* - 6\epsilon_V^* \\ & n_T \simeq -2\epsilon_V^* \\ & r \simeq 16\epsilon_V^* \\ & \hookrightarrow r = -8 n_T \text{ consistency condition} \end{aligned}}$$

*: when CMB modes exited horizon

<u>Worked example</u>

$m^2\phi^2$ inflation driven by standard mass term of scalar field

$V(\phi) = \frac{1}{2} m^2 \phi^2 \qquad V'(\phi) = m^2\phi \qquad V''(\phi) = m^2$

$$\epsilon_V(\phi) = \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right) = 2\left(\frac{M_{Pl}}{\phi}\right)^2 \qquad \eta_V(\phi) = M_{Pl}^2 \frac{V''}{V} = 2\left(\frac{M_{Pl}}{\phi}\right)^2 = \epsilon_V(\phi)!$$

$\epsilon_V, |\eta_V| < 1 \Longrightarrow \phi \gtrsim \sqrt{2} M_{Pl}$ inflation ends when $\phi \approx \sqrt{2} M_{Pl}$ $(\epsilon_V \approx 1, |\eta_V| \approx 1)$

$$N(\phi) = \int_{\phi_{end}}^{\phi} \frac{d\phi'}{\sqrt{2\epsilon_V(\phi')}} = \int_{\phi_{end}}^{\phi} \frac{d\phi'}{2M_{Pl}} \phi' = \int_{\sqrt{2}M_{Pl}}^{\phi} \frac{d\phi' \phi'}{2M_{Pl}} = \frac{\phi'^2}{4M_{Pl}}\Bigg|_{\sqrt{2}M_{Pl}}^{\phi} = \frac{\phi^2}{4M_{Pl}^2} - \frac{1}{2}$$

$$N(\phi_{CMB}) \sim 60 \Rightarrow \frac{\phi^2}{4M_{Pl}^2} - \frac{1}{2} \simeq 60 \rightarrow \phi_{CMB} \sim 2\sqrt{N_{CMB}}\, M_{Pl} \sim 15 M_{Pl}$$

so fluctuations observed in the CMB are created when ϕ is well super-Planckian

$$\phi_{CMB} = 2\sqrt{N_{CMB}}\, M_{Pl} \Rightarrow \varepsilon_V^* = \eta_V^* = 2\left(\frac{M_{Pl}}{\phi_{CMB}}\right)^2 = \frac{1}{2N_{CMB}}$$

$$n_s = 1 + 2\eta_V^* - 6\varepsilon_V^* \simeq 1 - \frac{2}{N_{CMB}} \simeq 0.96$$

in excellent agreement with observations

$n_s \sim 0.965 \pm 0.004$ (Planck 2018)

$$r = 16\,\varepsilon_V = \frac{8}{N_{CMB}} \approx 0.1$$

excluded by observations

$r \lesssim 0.07$ (Planck + BICEP)

(Alternative clearer presentation of horizon and flatness problem)

- $$\eta = \int_0^a \frac{da'}{Ha'^2} \propto \begin{cases} a & RD \longrightarrow a(t) \propto t^{1/2}, \quad (aH)^{-1} \propto t^{1/2} \\ a^{1/2} & MD \longrightarrow a(t) \propto t^{2/3}, \quad (aH)^{-1} \propto t^{1/3} \end{cases}$$

η grows monotonically with time so at any given time a scale entering the horizon was super-horizon when the CMB formed

- $$H^2 = \frac{\rho(a)}{3} - \frac{k}{a^2} \qquad (8\pi G = 1)$$

$$\Rightarrow 1 - \Omega(a) = -\frac{k}{(aH)^2} \qquad \Omega - 1 = \frac{\rho - \rho_{crit}}{\rho_{crit}}$$

~~In MD, RD~~ But $\frac{1}{(aH)^2}$ always grows in MD, RD

So $\Omega \sim 1$ today requires extreme fine-tuning in the past!