Scalar field implementation of inflation

Can a scalar field $\phi(\bar{x},t)$ give $P<0$?

$$T^{\alpha}{}_{\beta} = g^{\alpha\nu}\frac{\partial\phi}{\partial x^{\nu}}\frac{\partial\phi}{\partial x^{\beta}} - g^{\alpha}{}_{\beta}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x^{\nu}} + V(\phi)\right]$$

$(-+++)$ signature

$\left[V(\phi) = \frac{1}{2}m^2\phi^2 \text{ for free field}\right]$

$$\phi = \underbrace{\phi^0(t)}_{\text{homogeneous part}} + \underbrace{\delta\phi(\bar{x},t)}_{\text{1st order perturbation}}$$

Let's first just look at the homogeneous part to see how it affects the evolution of the scale factor

$$\phi \approx \phi^0(t) \longrightarrow \frac{\partial\phi}{\partial x^i} = 0$$

$\cancel{T^{(0)\alpha}{}_{\beta} = -g^{\alpha}}$ $\qquad T^{(0)\,0}{}_{0} = -\rho \qquad T^{(0)\,i}{}_{i} = P$

$$T^{(0)\,0}{}_{0} = g^{0\nu}\frac{\partial\phi}{\partial x^{\nu}}\frac{\partial\phi}{\partial x^{0}} - g^{0}{}_{0}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x^{\nu}} + V(\phi)\right]$$

$$= g^{00}\left(\frac{\partial\phi}{\partial x^0}\right)^2 - g^{0}{}_{0}\left[\frac{1}{2}g^{00}\left(\frac{\partial\phi}{\partial x^0}\right)^2 + V(\phi)\right] =$$

$\nu = 0$
$\mu = \nu = 0$
$x^{\nu} = -t$
$g^{00} = -1,\ g^{0}{}_{0} = 1$

$$= -\left(\frac{d\phi^{(0)}}{dt}\right)^2 - \left[-\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)\right] = -\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi) = -\rho$$

$$\Longrightarrow \rho = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)$$

$$T^{(0)\,i}{}_{i} = \cancel{g^{i}{}_{\nu}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^{\nu}}}^{\,0} - g^{i}{}_{i}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x^{\nu}} + V(\phi)\right] = -g^{i}{}_{i}\left[-\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)\right] =$$

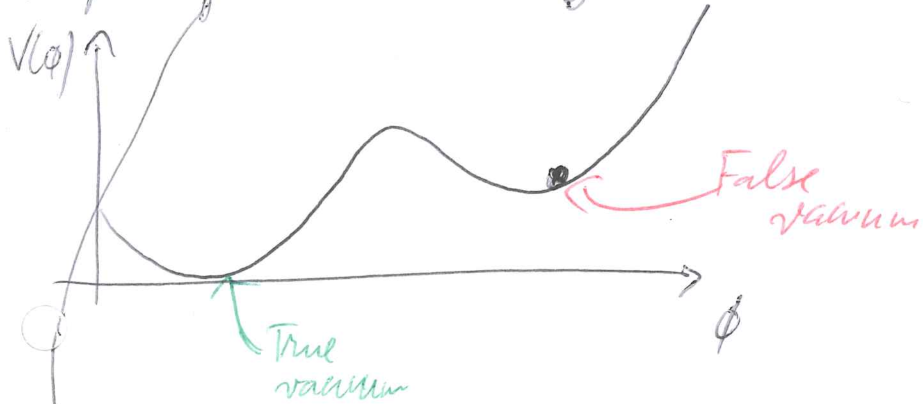
$$\xrightarrow{g^{i}{}_{i} = 1} = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi) = P \longrightarrow P = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi)$$

$$\rho = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)$$

$$P = \frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 - V(\phi)$$

Field configuration with more potential than kinetic energy: negative pressure

Example: field trapped in false vacuum



$\phi^{(0)} \simeq$ const so $\rho \simeq V(\phi)$, $P \simeq -V(\phi)$ $\rightarrow w = -1$

$\rho \simeq V(\phi) = \text{const}$ $\rightarrow$ very different from matter (a^{-3}), radiation (a^{-4})

behaves like Λ!

An universe with even a bit of false vacuum energy will quickly be dominated by it

Scale factor evolution

$$H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho = \text{const} \rightarrow \frac{1}{a}\frac{da}{dt} = H = \text{const}$$

$$\int \frac{da}{a} = H \int dt \qquad \ln a = Ht + \text{const} \rightarrow a \propto e^{Ht} \propto e^{\sqrt{\rho}\, t} \propto e^{\sqrt{V(\phi)}\, t} \propto e^{\sqrt{V_{FV}}\, t}$$

Primordial comoving horizon $\eta_{prim} = \int_{t_e}^{t_b} \frac{dt'}{a(t')}$

$t_b \rightarrow$ beginning of inflation

$t_e \rightarrow$ end of inflation

$$\eta_{prim} = \int_{t_e}^{t_b} \frac{dt}{a_e e^{H(t-t_b)}} = \frac{1}{a_e}\int_{t_e}^{t_b} \frac{dt}{e^{H(t-t_b)}} = -\frac{1}{Ha_e} e^{-H(t-t_b)}\Big|_{t_e}^{t_b} =$$

$$= -\frac{1}{Ha_e}\left(e^{-H(t_b-t_b)} - e^{-H(t_e-t_b)}\right) = \frac{1}{Ha_e}\left(e^{H(t_e-t_b)} - 1\right)$$

$$\eta_{prim} = \frac{1}{Ha_e}\left(e^{H(t_e - t_s)} - 1\right) \quad \text{requires} \quad H(t_e - t_s) > 60$$

(more than 60 e-foldings of inflation)

Nowadays most popular models based on field rolling down its potential, otherwise ~~regions of~~ "bubbles" of true vacuum never coalesce in time

Consider generic equations for scale factor when $V(\phi) \neq \text{const}$

Recall two Friedmann equations

$$H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho \simeq \frac{8\pi G}{3}\left[\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)\right]$$

$$\frac{1}{a}\frac{d^2a}{dt^2} + \frac{1}{2}\left(\frac{1}{a}\frac{da}{dt}\right)^2 = -4\pi G P \implies \frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(3P + \rho)$$

Take derivative of 1st Friedmann equation

$$\frac{d}{dt}(H^2) = 2H\frac{dH}{dt} \qquad \frac{dH}{dt} = \frac{d}{dt}\left(\frac{da/dt}{a}\right) = \frac{d^2a/dt^2\, a - (da/dt)^2}{a^2} =$$

$$= \frac{1}{a}\frac{d^2a}{dt^2} - \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{1}{a}\frac{d^2a}{dt^2} - H^2$$

$$\frac{d}{dt}(H^2) = 2H\frac{dH}{dt} = 2\frac{da/dt}{a}\left[\frac{d^2a/dt^2}{a} - \left(\frac{da/dt}{a}\right)^2\right]$$

$$\frac{d}{dt}(H^2) = \frac{d}{dt}\left(\frac{8\pi G}{3}\rho\right) = \underset{\times\frac{8\pi G}{3}}{\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)^2 + V(\phi)\right]} = \underset{\times\frac{8\pi G}{3}}{\left[\underbrace{2\,\frac{1}{2}\left(\frac{d\phi^{(0)}}{dt}\right)\left(\frac{d^2\phi^{(0)}}{dt^2}\right)}_{\text{chain rule}} + \frac{dV(\phi)}{dt}\right]} =$$

$$= \frac{8\pi G}{3}\left[\left(\frac{d\phi^{(0)}}{dt}\right)\left(\frac{d^2\phi^{(0)}}{dt^2}\right) + \underbrace{\frac{dV(\phi)}{d\phi^{(0)}}\frac{d\phi^{(0)}}{dt}}_{\text{chain rule}}\right] = \frac{8\pi G}{3}\left[\left(\frac{d\phi^{(0)}}{dt}\right)\left(\frac{d^2\phi^{(0)}}{dt^2}\right) + V'\frac{d\phi^{(0)}}{dt}\right]$$

$V' = dV(\phi)/d\phi^{(0)}$

Putting everything together

$$\frac{2\,da/dt}{a}\Bigg[\underbrace{\frac{d^2a/dt^2}{a}}_{\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho+3P)} - \underbrace{\left(\frac{da/dt}{a}\right)}_{\left(\frac{1}{a}\frac{da}{dt}\right)^2 = H^2 = \frac{8\pi G}{3}\rho}\Bigg]^2 = \frac{8\pi G}{3}\left[\left(\frac{d\phi^{(0)}}{dt}\right)\left(\frac{d^2\phi^{(0)}}{dt^2}\right) + V'\frac{d\phi^{(0)}}{dt}\right]$$

LHS

$$\Rightarrow \frac{2\,da/dt}{a}\left[-\frac{4\pi G}{3}(\rho+3P) - \frac{8\pi G}{3}\rho\right] = 8\pi G\,\frac{da/dt}{a}\left[-\frac{\rho}{3} - P - \frac{2\rho}{3}\right] =$$

$$= -8\pi G\,\frac{da/dt}{a}(\rho+P) = -8\pi G\,\frac{da/dt}{a}\left(\frac{d\phi^{(0)}}{dt}\right)^2 = -8\pi G H\left(\frac{d\phi^{(0)}}{dt}\right)^2$$

$\rho = \frac{1}{2}\dot\phi^2 + V(\phi)$

$P = \frac{1}{2}\dot\phi^2 - V(\phi)$ $\Rightarrow \rho + P = \dot\phi^2$

LHS=RHS

$$\Rightarrow -8\pi G H\left(\frac{d\phi^{(0)}}{dt}\right)^2 = \frac{8\pi G}{3}\cancel{\left(\frac{d\phi^{(0)}}{dt}\right)}\left(\frac{d^2\phi^{(0)}}{dt^2}\right) + V'\cancel{\frac{d\phi^{(0)}}{dt}}$$

$$\Rightarrow \boxed{\frac{d^2\phi^{(0)}}{dt^2} + 3H\frac{d\phi^{(0)}}{dt} + V' = 0}$$

Evolution of homogeneous scalar field in an expanding Universe

More useful in terms of conformal time

$$d\eta = \frac{dt}{a} \Rightarrow \frac{d}{d\eta} = a\frac{d}{dt} \Rightarrow \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta} \qquad \frac{d}{dt} = \frac{\dot{\ }}{a} \qquad \frac{df}{dt} = \frac{\dot f}{a}$$

$$H = \frac{1}{a}\frac{da}{dt} = \frac{1}{a^2}\frac{da}{d\eta} = \frac{\dot a}{a^2}$$

$$\frac{d^2}{dt^2} = \frac{1}{a}\frac{d}{d\eta}\left(\frac{1}{a}\frac{d}{d\eta}\right) = \frac{1}{a^2}\frac{d^2}{d\eta^2} - \frac{1}{a}\frac{\dot a}{a^2}\frac{d}{d\eta} = \frac{1}{a^2}\frac{d^2}{d\eta^2} - \frac{\dot a}{a^3}\frac{d}{d\eta} = \frac{1}{a^2}\frac{d^2}{d\eta^2} - \frac{H}{a}\frac{d}{d\eta}$$

Symbolically: $\frac{d^2}{dt^2} = \frac{\ddot{\ }}{a^2} - \frac{H\dot{\ }}{a}$ $\qquad \frac{d^2f}{dt^2} = \frac{\ddot f}{a^2} - \frac{H\dot f}{a}$

$$\Rightarrow \frac{d^2\phi^{(0)}}{dt^2} + 3H\frac{d\phi^{(0)}}{dt} + V' = \frac{\ddot\phi^{(0)}}{a^2} - \frac{H\dot\phi^{(0)}}{a} + \frac{3H\dot\phi^{(0)}}{a} + V' = 0$$

$$\Rightarrow \boxed{\ddot{\phi}^{(0)} + 2aH\dot{\phi}^{(0)} + a^2V' = 0}$$

Most inflationary models are slow-roll models

↳ $\phi^{(0)}$ and therefore H vary slowly

$$\eta \equiv \int_{t_e}^{t} \frac{dt'}{a(t')} = \int_{a_e}^{a} \frac{dt'}{da'}\frac{da'}{a'(t')} = \int_{a_e}^{a} da' \left(\frac{1}{a'}\frac{da'}{dt'}\right)^{-1}\frac{1}{a'} = \int_{a_e}^{a}\frac{da'}{a'^2 H} =$$

negative! $t < t_e$

$$= \int_{a_e}^{a} da \frac{1}{Ha^2} \simeq \frac{1}{H}\int_{a_e}^{a}\frac{da}{a^2} = \frac{1}{H}\left[-\frac{1}{a}\right]\Big|_{a_e}^{a} = \frac{1}{H}\left(-\frac{1}{a}+\frac{1}{a_e}\right) \simeq -\frac{1}{aH}$$

abuse of notation

$H \simeq$ const

$a_e \gg a$

$$\eta \simeq -\frac{1}{aH}$$

Two variables usually adopted to quantify slow-roll

$$\varepsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{1}{H^2}\frac{dH}{dt} = -\frac{1}{H^2}\frac{1}{a}\frac{dH}{d\eta} = -\frac{\dot{H}}{aH^2}$$

$\left[RD \quad a \propto \eta \quad H = \frac{1}{\eta}\right]$

$\varepsilon \longrightarrow 0$ as $\phi, H \longrightarrow$ const

H always decreasing $\longrightarrow \varepsilon > 0$ $\quad$ $\varepsilon \ll 1$ during inflation ($\varepsilon < 1$ definition of inflation)

$$\delta \equiv \frac{1}{H}\frac{d^2\phi^{(0)}/dt^2}{d\phi^{(0)}/dt} = \frac{1}{H\left(\frac{1}{a}\dot{\phi}^{(0)}\right)}\left[\frac{\ddot{\phi}^{(0)}}{a^2} - \frac{H}{a}\dot{\phi}^{(0)}\right] = -\frac{a}{H\dot{\phi}^{(0)}}\left[\frac{H}{a}\dot{\phi}^{(0)} - \frac{\ddot{\phi}^{(0)}}{a^2}\right] =$$

$$= -\frac{1}{aH\dot{\phi}^{(0)}}\left[aH\dot{\phi}^{(0)} - \ddot{\phi}^{(0)}\right] = -\frac{1}{aH\dot{\phi}^{(0)}}\left[3aH\dot{\phi}^{(0)} + a^2V'\right]$$

$-\ddot{\phi}^{(0)} = 2aH\dot{\phi}^{(0)} + a^2V'$

$\delta \ll 1$ during inflation

End of inflation is typically defined as moment when $\varepsilon \simeq 1$

Inflationary gravitational waves

- Inflation solves horizon problem and correlates otherwise disconnected scales, and when these were causally connected it also generates (scalar and tensor) perturbations

scalar → couple to matter density radiation → inhomogeneities and anisotropies

tensor → fluctuations to the gravitational metric → GWs! (not coupled to density)

- Start by looking at tensor perturbations as they are much simpler (scalar fluctuations couple to energy density fluctuations)

Idea: QM fluctuations during inflation generate the scalar and tensor perturbations we can observe today

$$\phi^{(0)} + \delta\phi \qquad g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$$

$$\langle \delta\phi \rangle = 0$$

non-zero variance! → "$\langle \delta\phi^2 \rangle - \langle \delta\phi \rangle^2$" $\neq 0$ "$\langle \delta g_{\mu\nu}^2 \rangle - \langle \delta g_{\mu\nu} \rangle^2$" $\neq 0$

↳ use to set initial conditions generated by inflation

Quantization of simple harmonic oscillator

Let's start from a simpler example (we will try to write more complicated examples in this form)

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \xrightarrow{\text{quantization}} \hat{x} = v(\omega,t)\,\hat{a} + v^*(\omega,t)\,\hat{a}^\dagger$$

quantization: $x \longrightarrow \hat{x}$ quantum operator

v solves $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$

so $v \propto e^{-i\omega t}$

acts on states of system → $\hat{a}$ quantum operator

$$\hat{a}|0\rangle = 0$$

vacuum state

$$[\hat{a},\hat{a}^\dagger] \equiv \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \qquad [\hat{a},\hat{a}] = [\hat{a}^\dagger,\hat{a}^\dagger] = 0$$

$[\hat{x},\hat{p}] = i$ where $p = \frac{dx}{dt}$ (if v normalized as $v(\omega,t) = \frac{e^{-i\omega t}}{\sqrt{2\omega}}$)

Look at quantum fluctuations of operator $\hat{x}$ in vacuum

$$\langle|\hat{x}|^2\rangle \equiv \langle 0|\hat{x}^\dagger x|0\rangle = \langle 0|(v^*\hat{a}^\dagger + \bar{v}\hat{a})(v\hat{a} + v^*\hat{a}^\dagger)|0\rangle =$$

$$= \cancel{\langle 0|v^*\hat{a}^\dagger \hat{a}|0\rangle}_{\hat{a}|0\rangle=0} + \cancel{\langle 0|v^*\hat{a}^\dagger v^*\hat{a}^\dagger|0\rangle} + \cancel{\langle 0|v\hat{a}v\hat{a}|0\rangle}_{\hat{a}|0\rangle=0}$$

$$\langle 0|\hat{a}^\dagger = (\hat{a}|0\rangle)^\dagger = 0$$

$$\hat{a}\hat{a}^\dagger = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a} = [\hat{a},\hat{a}^\dagger] + \hat{a}^\dagger\hat{a}$$

$$+ \langle 0|v\hat{a}\, v^*\hat{a}^\dagger|0\rangle = |v(\omega,t)|^2\langle 0|\hat{a}\hat{a}^\dagger|0\rangle =$$

$$= |v(\omega,t)|^2\langle 0|[\hat{a},\hat{a}^\dagger] + \cancel{\hat{a}^\dagger\hat{a}}_{\hat{a}|0\rangle=0}|0\rangle = |v(\omega,t)|^2\langle 0|1|0\rangle = |v(\omega,t)|^2$$

$$\Longrightarrow \langle|\hat{x}|^2\rangle = |v(\omega,t)|^2 = \left|\frac{e^{-i\omega t}}{\sqrt{2\omega}}\right|^2 = \frac{e^{-i\omega t}}{\sqrt{2\omega}}\frac{e^{i\omega t}}{\sqrt{2\omega}} = \frac{1}{2\omega}$$

non-zero variance!

Similar calculation for tensor perturbations

Tensor perturbations

Recall evolution equations for tensor perturbations

$$\begin{cases} \ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + K^2 h_+ = 0 \\ \ddot{h}_\times + 2\frac{\dot{a}}{a}\dot{h}_\times + K^2 h_\times = 0 \end{cases}$$

$\rightarrow$ not exactly simple harmonic oscillator, but we want to bring it into that form

Redefinition $\tilde{h} \equiv \frac{a h}{\sqrt{16\pi G}}$ $\qquad h = h_+, h_\times$

Take derivatives with respect to η

$$\frac{\dot h}{\sqrt{16\pi G}} = \frac{d}{d\eta}\left(\frac{\tilde h}{a}\right) = \frac{\dot{\tilde h}}{a} - \frac{\dot a}{a^2}\tilde h \qquad \left[\text{note: } \frac{\dot a}{a^2}\tilde h = \frac{1}{a}\frac{\dot a}{a}h = \frac{\mathcal{H}}{a}h = Hh\right]$$

$$\frac{\ddot h}{\sqrt{16\pi G}} = \frac{d}{d\eta}\left[\frac{\dot{\tilde h}}{a} - \frac{\dot a}{a^2}\tilde h\right] = \frac{\ddot{\tilde h}}{a} - \frac{\dot a}{a^2}\dot{\tilde h} - \tilde h\frac{d}{d\eta}\left(\frac{\dot a}{a^2}\right) - \frac{\dot a}{a^2}\dot{\tilde h} =$$

$$\frac{d}{d\eta}\left(\frac{\dot a}{a^2}\right) = \frac{\ddot a}{a^2} - \frac{2(\dot a)^2}{a^3}$$

$$= \frac{\ddot{\tilde h}}{a} - \frac{\dot a}{a^2}\dot{\tilde h} - \frac{\ddot a}{a^2}\tilde h + \frac{2(\dot a)^2}{a^3}\tilde h - \frac{\dot a}{a^2}\dot{\tilde h} =$$

$$= \frac{\ddot{\tilde h}}{a} - 2\frac{\dot a}{a^2}\dot{\tilde h} - \frac{\ddot a}{a^2}\tilde h + \frac{2(\dot a)^2}{a^3}\tilde h = \frac{\ddot{\tilde h}}{a} - 2\frac{\dot a}{a^2}\dot{\tilde h} + \left[\frac{2(\dot a)^2}{a^3} - \frac{\ddot a}{a^2}\right]\tilde h =$$

$$= \frac{\ddot{\tilde h}}{a} - 2H\tilde h + \left[\frac{2(\dot a)^2}{a^3} - \frac{\ddot a}{a^2}\right]\tilde h$$

Let's plug these into the equation for h

$$\ddot h + 2\frac{\dot a}{a}\dot h + k^2 h = 0 \Longrightarrow \frac{\ddot h}{\sqrt{16\pi G}} + 2\frac{\dot a}{a}\frac{\dot h}{\sqrt{16\pi G}} + k^2\underbrace{\frac{h}{\sqrt{16\pi G}}}_{=\frac{\tilde h}{a}} = 0$$

$$\Longrightarrow \frac{\ddot{\tilde h}}{a} - 2\frac{\dot a}{a^2}\dot{\tilde h} - \frac{\ddot a}{a^2}\tilde h + \frac{2(\dot a)^2}{a^3}\tilde h + 2\frac{\dot a}{a}\frac{\dot{\tilde h}}{a} - 2\frac{\dot a}{a}\frac{\dot a}{a^2}\tilde h + k^2\frac{\tilde h}{a} =$$

$$= \frac{\ddot{\tilde h}}{a} - \cancel{2\frac{\dot a}{a^2}\dot{\tilde h}} - \frac{\ddot a}{a^2}\tilde h + \cancel{\frac{2(\dot a)^2}{a^3}\tilde h} + \cancel{2\frac{\dot a}{a^2}\dot{\tilde h}} - \cancel{\frac{2(\dot a)^2}{a^3}\tilde h} + k^2\frac{\tilde h}{a} =$$

$$= \frac{\ddot{\tilde h}}{a} + k^2\frac{\tilde h}{a} - \frac{\ddot a}{a^2}\tilde h = \boxed{\frac{1}{a}\left[\ddot{\tilde h} + \left(k^2 - \frac{\ddot a}{a}\right)\tilde h\right] = 0}$$

It is of the simple harmonic oscillator form

Recall $\frac{d^2x}{dt^2} + \omega^2 x = \ddot x + \omega^2 x = 0$ (no damping term $\propto \dot{\tilde h}$)

So we can immediately write: $\boxed{\hat{\tilde h}(\bar k,\eta) = v(k,\eta)\hat a_{\bar k} + v^*(k,\eta)\hat a^{\dagger}_{\bar k}}$

where v solves $\boxed{\ddot v + \left(k^2 - \frac{\ddot a}{a}\right)v = 0}$

Since for the simple harmonic oscillator

$$\langle|\hat{x}|^2\rangle = |v(\omega,t)|^2$$

We can write

equal-time correlator

$$\langle|\tilde{h}|^2\rangle \equiv \langle \hat{\tilde{h}}^{+}(\bar{k},\eta)\,\hat{\tilde{h}}(\bar{k}',\eta)\rangle = |v(\bar{k},\eta)|^2 (2\pi)^3 \delta^3(\bar{k}-\bar{k}')$$

The main difference compared to the simple harmonic oscillator is the factor $(2\pi)^3\delta^3(\bar{k}-\bar{k}')$ because a quantum field can be thought of as a ~~collection~~ collection of infinite simple harmonic oscillators, one for each $\bar{k}$, which are independent (uncorrelated) if the equations are linear

Since $\tilde{h} = \dfrac{ah}{\sqrt{16\pi G}} \longrightarrow h = \dfrac{\sqrt{16\pi G}}{a}\tilde{h}$

$$\langle \hat{h}^{+}(\bar{k},\eta)\,\hat{h}(\bar{k}',\eta)\rangle = \frac{16\pi G}{a^2}|v(k,\eta)|^2(2\pi)^3\delta^3(\bar{k}-\bar{k}') \equiv (2\pi)^3 P_h(k)\,\delta^3(\bar{k}-\bar{k}')$$

$$P_h(k) \equiv \frac{16\pi G}{a^2}|v(k,\eta)|^2$$ POWER SPECTRUM

(dimensionless power spectrum $\Delta(k) \equiv \dfrac{k^3 P(k)}{2\pi^2}$)

→ quantifies variance/strength of fluctuations on a certain scale!

So now we just need to solve for $v(k,\eta)$ during inflation

$$\ddot{v} + \left(k^2 - \frac{\ddot{a}}{a}\right)v = 0$$

During inflation $\eta \simeq -\dfrac{1}{aH}$ $\left(= \displaystyle\int_{a_e}^{a} \frac{da}{Ha^2}\Big|_{a_e \gg a,\ H\simeq \text{const}} \right)$

Recall $d\eta = \dfrac{dt}{a} \to \dfrac{d}{dt} = \dfrac{1}{a}\dfrac{d}{d\eta} \to H = \dfrac{1}{a}\dfrac{\dot{a}}{a} = \dfrac{\dot{a}}{a^2}$ ~~$\ddot{a} = a^2 H$~~

and also $H \simeq -\dfrac{1}{a\eta}$ $\Big\}$ $\dfrac{\dot{a}}{a^2} = -\dfrac{1}{a\eta} \Rightarrow \boxed{\dot{a} = -\dfrac{a}{\eta}}$

(only during inflation!)

$\Rightarrow$ ~~$\dfrac{\ddot{a}}{a} = \dfrac{1}{a}\dfrac{d}{d\eta}(\dot{a}) = \dfrac{1}{a}\dfrac{d}{d\eta}\left(-\dfrac{a}{\eta}\right) = \dfrac{1}{a}\left(-\dfrac{\dot{a}}{\eta^2} + \dfrac{a}{\eta^2}\right)$~~