INHOMOGENEITIES

- Now we want to explicitly solve the Einstein-Boltzmann system with initial conditions provided by inflation

Recall what is the system and initial conditions, neglecting polarization and assuming massless neutrinos

(recall also $\delta_\gamma \sim 4\Theta_0$, $\delta_\nu \sim 4\mathcal{N}_0$, $v_\gamma \sim -3i\Theta_1$, $v_\nu \sim -3i\mathcal{N}_1$)

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_b\right] \quad \Big\}\ \cancel{DM}\ \gamma$$

$\dot{\tau} = -n_e \sigma_T a$

$$\Theta_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}\mathcal{P}_\ell(\mu)\Theta(\mu)$$

$$\mathcal{N}_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}\mathcal{P}_\ell(\mu)\mathcal{N}(\mu)$$

- $$\left.\begin{aligned}\dot{\delta}_{dm} + ikv_{dm} &= -3\dot{\Phi}\\ \dot{v}_{dm} + \frac{\dot{a}}{a}v_{dm} &= -ik\Psi\end{aligned}\right\}\ DM$$

$\dot{} \equiv \frac{d}{d\eta} = a\frac{d}{dt}$

$$\left.\begin{aligned}\dot{\delta}_b + ikv_b &= -3\dot{\Phi}\\ \dot{v}_b + \frac{\dot{a}}{a}v_b &= -ik\Psi + \frac{\dot{\tau}}{R}\left[v_b + 3i\Theta_1\right]\end{aligned}\right\}\ b$$

$R = \frac{3\rho_b^{(0)}}{4\rho_\gamma^{(0)}}$

$$\dot{\mathcal{N}} + ik\mu\mathcal{N} = -\dot{\Phi} - ik\mu\Psi \quad \Big\}\ \nu$$

- $$K^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2\left(\rho_{dm}\delta_{dm} + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right)$$

$$K^2(\Phi + \Psi) = -32\pi G a^2\left(\rho_\gamma\Theta_2 + \rho_\nu\mathcal{N}_2\right)$$

$\Big\}$ Φ, Ψ scalar / curvature perturbations to the metric

$$\ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + K^2 h_+ = 0$$

$$\ddot{h}_\times + 2\frac{\dot{a}}{a}\dot{h}_\times + K^2 h_\times = 0$$

$\Big\}$ $h_+, h_\times$ tensor perturbation to the metric / GWs

$$\Phi(k,\eta_i) = -\Psi(k,\eta_i) = 2\Theta_0(k,\eta_i) = 2\mathcal{N}_0(k,\eta_i)$$

$$\delta_{dm}(k,\eta_i) = \delta_b(k,\eta_i) = 3\Theta_0(k,\eta_i) = \frac{3}{2}\Phi(k,\eta_i) = -\frac{3}{2}\Psi(k,\eta_i)$$

- $$\Theta_1(k,\eta_i) = \mathcal{N}_1(k,\eta_i) = \frac{iv_{dm}(k,\eta_i)}{3} = \frac{iv_b(k,\eta_i)}{3} = -\frac{k\Phi(k,\eta_i)}{6aH}$$

η_i such that $k\eta_i \ll 1$ for all modes of interest, but well after inflation ended

Our goal: figure out $\underbrace{\text{inhomogeneities}}_{\text{late-Universe matter}}$ and $\underbrace{\text{anisotropies}}_{\text{CMB}}$

We start from inhomogeneities, looking at evolution of perturbations to the DM. Simple picture:

Early times	Late times
Radiation ~ $\Theta_0, \Theta_1, N_0, N_1$	Radiation negligible
$\downarrow$	
affects Φ, Ψ	Φ, Ψ
$\downarrow$	$\downarrow$
affects δ, v (indirectly)	affect δ, v

We want to solve for the evolution of each Fourier mode $\delta_{dm}(k,\eta)$

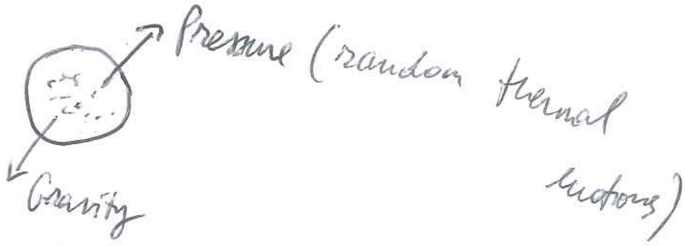
[from now on $\underline{\delta_{dm} \longrightarrow \delta}$]

Knowing $\delta(k,\eta)$, with initial conditions generated by inflation, we can compute (dark) matter power spectrum today, compare to observations (at least on large scales)

<u>Solution strategy</u>

Basic idea of gravitational instability [sub-horizon]

$$\ddot{\delta} + [\text{Pressure} - \text{Gravity}]\,\delta = 0$$



<u>Any</u> initial overdensity will eventually grow

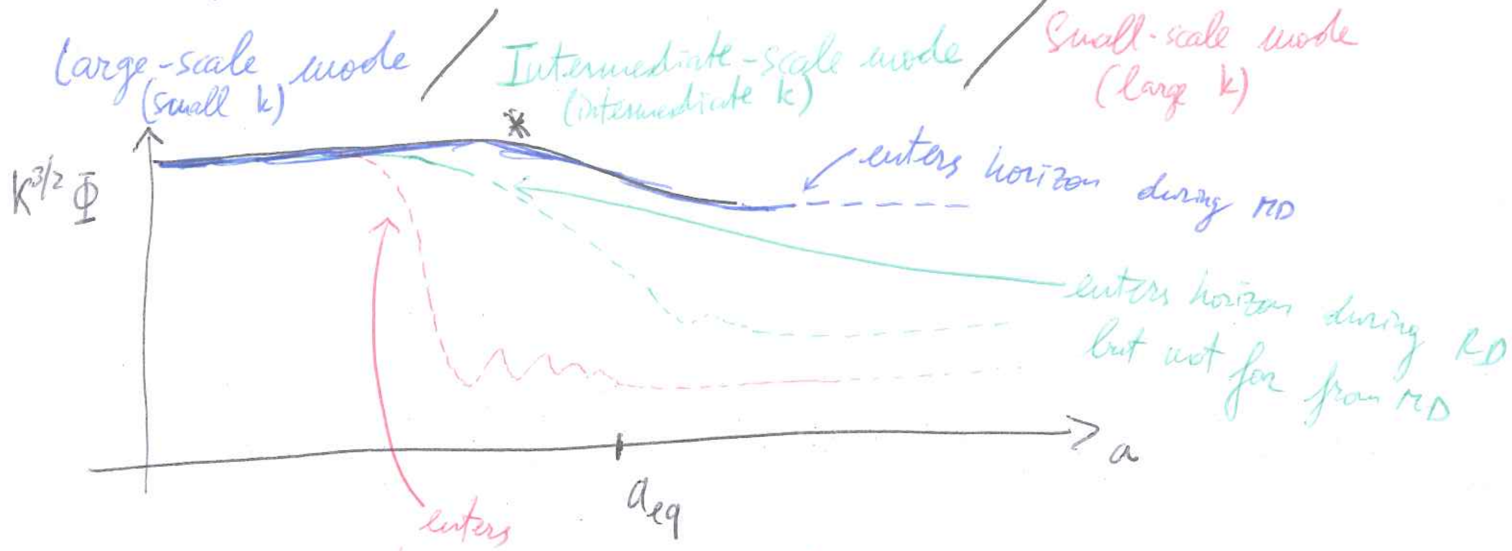
Schematically:

- large P, δ oscillates
- low P, δ grows (exponentially) $\longrightarrow$ actually $\begin{cases} \delta \propto t & \text{MD} \\ \delta \propto \ln(t) & \text{RD} \end{cases}$

Three natural stages of evolution (a posteriori)

- Early times: mode is super-horizon, $\Phi \sim$ const ($k\eta \ll 1$)
- Intermediate times: mode enters horizon and radiation-to-matter domination $\rightarrow$ what happens to the mode depends on whether it enters during RD (damped oscillations) or MD ($\sim$ const). Larger scale modes enter the horizon later (during MD rather than RD) so we expect suppression due to radiation pressure on small scales
- Late times: matter domination, $\Phi \sim$ const

Example, schematic plot



Schematically

$$\Phi(k,a) = \underbrace{\Phi_p(k)}_{\text{inflation}} \times \underbrace{\{\text{Transfer function}(k)\}}_{\substack{\text{evolution through horizon crossing}\\ \text{and matter domination}\\ \sim 1 \text{ on large scales}}} \times \underbrace{\{\text{Growth function}(a)\}}_{\substack{k\text{-independent}\\ \text{late-time growth}}}$$

- $T(k) \equiv \dfrac{\Phi(k, a_{late})}{\Phi_{large\ scale}(k, a_{late})}$ where a_{late} well in matter domination

$\Phi_{large\ scale}$ decreased by $\sim \frac{9}{10}$ relative to Φ_p *

Growth function D_1

$$\frac{D_1(a)}{a} \equiv \frac{\Phi(a)}{\Phi(a_{late})} \qquad (a > a_{late})$$

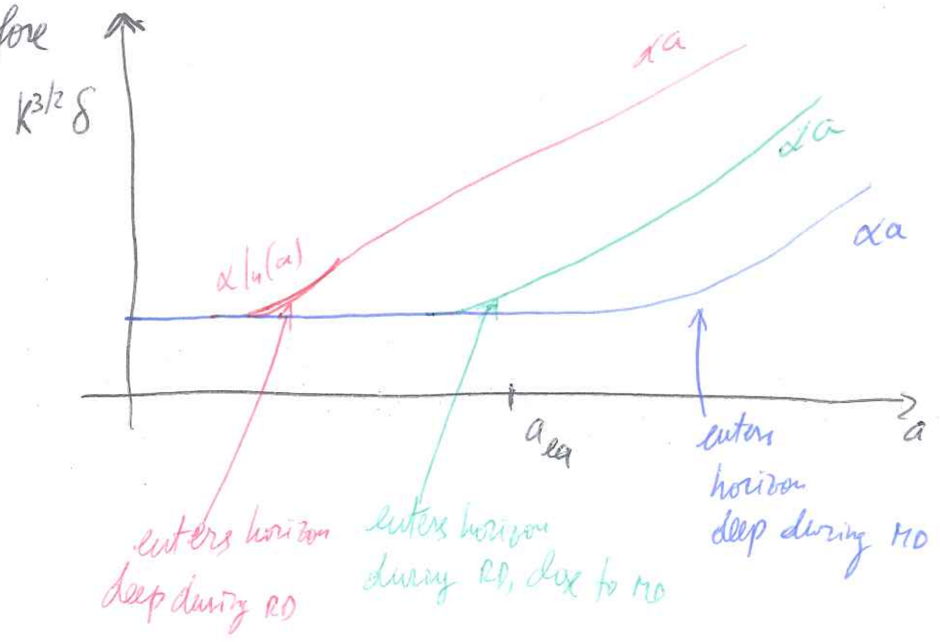
Because for $\Omega_\Lambda = 0, \Omega_m = 1$, $\Phi \sim$ const $\Rightarrow D_1 \propto a$

$$\Rightarrow \boxed{\Phi(\bar{k}, a) = \frac{9}{10}\Phi_p(\bar{k})\, T(k)\, \frac{D_1(a)}{a}} \qquad (a > a_{late})$$

How to probe Φ? Measuring matter distribution!

Late times: $\delta \propto D_1(a) \propto a$

Same modes as before



We want <u>matter</u> power spectrum $P = P_\delta$ in terms of Φ_p

Use Poisson equation

$$k^2\Phi = 4\pi G a^2\left[\rho_m\delta_m + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}\left(i\rho_m v_m + 4\rho_r\Theta_{r,1}\right)\right] \xrightarrow[a > a_{late}]{\text{large-}k \text{ no radiation}} \Phi \approx \frac{4\pi G\rho_m a^2\delta}{k^2}$$

$$\rho_m = \frac{\Omega_m\rho_{cr}}{a^3} \quad \text{and} \quad H_0^2 = \frac{8\pi G}{3}\rho_{cr} \rightarrow 4\pi G\rho_{cr} = \frac{3}{2}H_0^2$$

$$\Rightarrow \delta(\bar{k}, a) = \frac{k^2\Phi(\bar{k}, a)}{4\pi G\rho_m a^2} = \frac{k^2\Phi(\bar{k}, a)\, a^3}{\Omega_m\rho_{cr}\cancel{a^2}\, 4\pi G} = \frac{k^2\Phi(\bar{k}, a)\, a}{4\pi G\rho_{cr}\Omega_m} =$$

$$= \frac{k^2 \Phi(\bar{k},a)\, a}{3/2\, \Omega_m H_0^2} \implies \delta(\bar{k},a) \approx \frac{2}{3} \frac{k^2 \Phi(\bar{k},a)}{\Omega_m H_0^2} a \quad (a > a_{late})$$

But we know that $\Phi(\bar{k},a) = \frac{9}{10} \Phi_p(\bar{k}) T(k) \frac{D_1(a)}{a}$

$$\implies \delta(\bar{k},a) \approx \frac{2}{3} \frac{k^2 \Phi(\bar{k},a)}{\Omega_m H_0^2} a = \frac{\cancel{2}}{\cancel{3}} \frac{k^2 \cancel{a}}{\Omega_m H_0^2} \frac{\cancel{9}^{\,3}}{\cancel{10}_{\,5}} \Phi_p(\bar{k}) T(k) \frac{D_1(a)}{\cancel{a}} \approx$$

$$\implies \boxed{\delta(\bar{k},a) \approx \frac{3}{5} \frac{k^2}{\Omega_m H_0^2} \Phi_p(\bar{k}) T(k) D_1(a) \quad (a > a_{late})}$$

Regardless of how Φ_p was generated

Recall from inflation

$$P_\Phi(k) = \frac{50\pi^2}{9k^3} \left(\frac{k}{H_0}\right)^{n_s - 1} \delta_H^2 \left(\frac{\Omega_m}{D_1(a=1)}\right)^2$$

$\langle \Phi \rangle = 0$ $\qquad \langle \Phi^2 \rangle \sim P_\Phi(k)$

$$\implies P_\delta(k,a) = P_\delta = \frac{9}{25} \frac{k^4}{\Omega_m^2 H_0^4} T^2(k) D_1^2(a) P_\Phi(k) =$$

$$= \frac{\cancel{9}}{\cancel{25}} \frac{\cancel{k^4}}{\cancel{\Omega_m^2} H_0^4} T^2(k) D_1^2(a) \frac{\cancel{50}^{\,2}\pi^2}{\cancel{9k^3}} \frac{k^{n_s \cancel{-1}}}{H_0^{n_s - 1}} \delta_H^2 \frac{\cancel{\Omega_m^2}}{D_1^2(a=1)} = 2\pi^2 \delta_H^2 \frac{k^{n_s}}{H_0^{n_s+3}} T^2(k) \left(\frac{D_1(a)}{D_1(a=1)}\right)^2$$

$$\implies \boxed{P_\delta \approx 2\pi^2 \delta_H^2 \frac{k^{n_s}}{H_0^{n_s+3}} T^2(k) \left(\frac{D_1(a)}{D_1(a=1)}\right)^2 \quad (a > a_{late})}$$

$[P] = \text{length}^3$ $\quad [\text{Mpc}^3$, or typically, $h^{-3}\text{Mpc}^3]$

Multiply by k^3 to get dimensionless quantity

$$\frac{d^3k\, P(k)}{(2\pi)^3}$$: excess power in bin of width dk centered at k

$$\frac{d^3k\, P(k)}{(2\pi)^3} = \frac{dk\, k^2\, 4\pi}{8\pi^3} P(k) = dk\,\frac{k^2}{2\pi^2} P(k) = \frac{dk}{k}\,\frac{k^3 P(k)}{2\pi^2} =$$

$$= d(\ln k)\,\frac{k^3 P(k)}{2\pi^2} \equiv d(\ln k)\,\Delta^2(k)$$

$$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}$$ dimensionless power spectrum

What does Δ^2 measure?

$\Delta \ll 1$: small inhomogeneities

$\Delta \sim 1$: inhomogeneities becoming non-linear

$\Delta \gg 1$: large, non-linear inhomogeneities

On horizon-sized scale today $(k \sim H_0)$ Harrison-Zeldovich-Peebles $(n_s = 1)$ spectrum has

$$P(k, 1) = 2\pi^2\, \delta_H^2\, \frac{H_0^{1}}{H_0^{1+3}}\, \underbrace{T^2(k)}_{=1} \left(\cancel{\frac{D_1(1)}{D_1(1)}}\right)^2 = 2\pi^2 \frac{\delta_H^2}{H_0^3}$$

$$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2} = \frac{k^3}{2\pi^2}\, 2\pi^2\, \frac{\delta_H^2}{H_0^3} \underset{k = H_0}{=} \delta_H^2$$

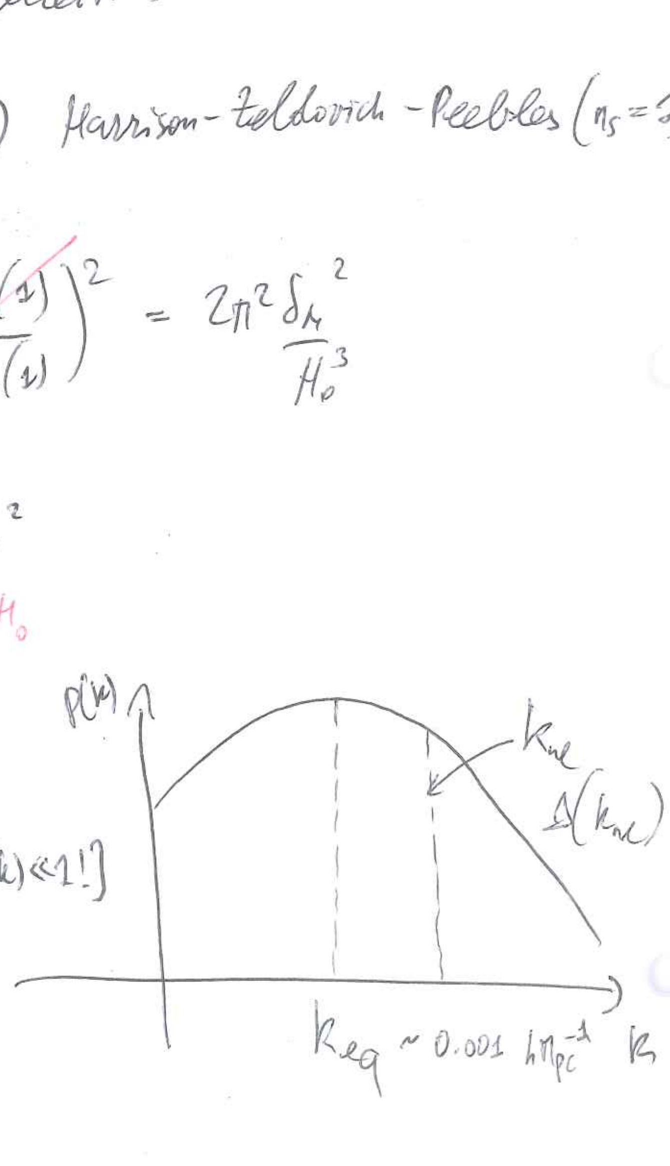
How do we expect $P(k)$ to be)

Large scales (small k) $T(k) = 1 \longrightarrow P(k) \sim k^{n_s} \sim k$

Small scales (large k) $\longrightarrow P(k)$ suppressed $[T(k) \ll 1!]$

modes enter horizon during RD

suppressed by the radiation pressure



Strategy

We can significantly reduce the full set of Einstein-Boltzmann equations

Simplifications:

- before recombination photons characterized by Θ_0, Θ_1 only!
- after recombination photons negligible (matter domination)

↳ we can neglect all radiation moments except monopole, dipole

Relevant equations

$$\boxed{\dot{\Theta}_r + ik\mu\Theta_r = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_{P,\gamma} - \Theta_r + \mu v_b\right]}$$

"r" = radiation
γ and ν

$$\dot{\delta} + ikv = -3\dot{\Phi}$$

$$\dot{v} + \frac{\dot{a}}{a}v = \cancel{ik\Phi} - ik\Psi$$

We can simplify these further, get equations for Θ_0 and Θ_1

- neglect baryons for the moment ~~(later)~~ $\longrightarrow \dot{\tau}[\ldots] = 0$
- $\Phi = -\Psi$ (fine as we neglect quadrupoles)
- integrate radiation equations to get Θ_0 and Θ_1 equations

$$\dot{\Theta} + ik\mu\Theta \simeq -\dot{\Phi} - ik\mu\Psi$$

$$\Theta_\ell = \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)$$

$$\Theta_0 = \int_{-1}^{1}\frac{d\mu}{2}\Theta(\mu) \qquad [P_0(\mu) = 1]$$

$$\Theta_1 = i\int_{-1}^{1}d\mu\frac{\mu}{2}\Theta(\mu) \qquad [P_1(\mu) = \mu]$$

$$\Theta_2 = \int_{-1}^{1}d\mu\left(\frac{1-3\mu^2}{4}\right)\Theta(\mu) \qquad \left[P_2(\mu) = \frac{3\mu^2-1}{2}\right]$$

$$\times\int\frac{d\mu}{2} \Rightarrow \frac{d\Theta}{d\eta} + ik\mu\Theta = -\frac{d\Phi}{d\eta} + ik\mu\Phi$$

$$\Rightarrow \int_{-1}^{+1}\frac{d\mu}{2}\frac{d\Theta}{d\eta} + ik\underbrace{\int_{-1}^{1}\frac{d\mu}{2}\mu\Theta}_{\Theta_1} = -\int_{-1}^{1}\frac{d\mu}{2}\frac{d\Phi}{d\eta} + ik\Phi\int_{-1}^{1}\frac{d\mu}{2}\mu$$

$$\Rightarrow \frac{d}{d\eta}\underbrace{\int_{-1}^{1}\frac{d\mu}{2}\Theta}_{\Theta_0} + k\Theta_1 = -\frac{d\Phi}{d\eta}\underbrace{\int_{-1}^{1}\frac{d\mu}{2}}_{1} + ik\Phi\cancelto{0}{\left.\frac{\mu^2}{4}\right|_{-1}^{1}}$$

$$\Rightarrow \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi}$$ monopole equation

$$\frac{d\Theta}{d\eta} + ik\mu\Theta = -\frac{d\Phi}{d\eta} + ik\mu\Phi \qquad \times\int\frac{d\mu}{2}\mu$$

$$\int_{-1}^{1}\frac{d\mu}{2}\mu\frac{d\Theta}{d\eta} + ik\int_{-1}^{1}\frac{d\mu}{2}\mu^2\Theta = -\int_{-1}^{1}\frac{d\mu}{2}\mu\frac{d\Phi}{d\eta} + ik\Phi\int_{-1}^{1}\frac{d\mu}{2}\mu^2$$

$$\Rightarrow \frac{d}{d\eta}\underbrace{\int_{-1}^{1}\frac{d\mu}{2}\mu\Theta}_{-i\Theta_1} + ik\int_{-1}^{1}\frac{d\mu}{2}\mu^2\Theta = -\frac{d\Phi}{d\eta}\cancel{\underbrace{\int_{-1}^{1}\frac{d\mu\,\mu}{2}}_{=\frac{\mu^2}{4}\Big|_{-1}^{1}=0}} + ik\Phi\frac{\mu^3}{6}\Big|_{-1}^{1}$$

$$\Rightarrow -i\frac{d\Theta_1}{d\eta} + ik\int_{-1}^{1}\frac{d\mu}{2}\mu^2\Theta = \frac{ik\Phi}{3} \qquad \times i$$

$$\Rightarrow \dot{\Theta}_1 - k\int d\mu\,\frac{\mu^2}{2}\Theta = -\frac{k\Phi}{3}$$

Evaluate approximately by demanding $\Theta_2 = 0$

$$\Theta_2 = \int_{-1}^{1} d\mu\left(\frac{1-3\mu^2}{4}\right)\Theta = \int_{-1}^{1}\frac{d\mu}{4}\Theta - \frac{3}{4}\int_{-1}^{1}d\mu\,\mu^2\Theta = \frac{1}{2}\underbrace{\int_{-1}^{1}\frac{d\mu}{2}\Theta}_{\frac{1}{2}\Theta_0} - \frac{3}{2}\underbrace{\int_{-1}^{1}d\mu\frac{\mu^2}{2}\Theta}_{\text{unknown}} \approx 0$$

$$\Rightarrow \frac{1}{2}\Theta_0 - \frac{3}{2}\int_{-1}^{1}d\mu\frac{\mu^2}{2}\Theta \approx 0 \Rightarrow \int_{-1}^{1}d\mu\frac{\mu^2}{2}\Theta \approx \frac{\cancel{2}}{3}\frac{1}{\cancel{2}}\Theta_0 = \frac{\Theta_0}{3}$$

$$\Rightarrow \dot{\Theta}_1 - \frac{k\Theta_0}{3} = -\frac{k\Phi}{3}$$

Putting everything together our Boltzmann equations simplify to

$$\dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi}$$

$$\dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi$$

$$\dot{\delta} + ikv = -3\dot{\Phi}$$

$$\dot{v} + \frac{\dot{a}}{a}v = ik\Phi \qquad (\Phi = -\Psi)$$

We also need equation for Φ

Two options (are redundant since we set $\Phi = -\Psi$):

- Time-time equation

$$k^2 \Phi + 3\frac{\dot a}{a}\left(\dot\Phi - \Psi\frac{\dot a}{a}\right) = 4\pi G a^2 \left(\rho_{dm}\delta + 4\rho_r \Theta_{r,0}\right)$$ [neglected baryons]

$$\Big\Downarrow \; \Phi = -\Psi$$

$$k^2 \Phi + 3\frac{\dot a}{a}\left(\dot\Phi + \frac{\dot a}{a}\Phi\right) = 4\pi G a^2 \left(\rho_{dm}\delta + 4\rho_r \Theta_{r,0}\right)$$

- Algebraic equation (exercise)

$$k^2 \Phi = 4\pi G a^2 \left[\rho_{dm}\delta + 4\rho_r \Theta_{r,0} + \frac{3aH}{k}\left(i\rho_{dm} v + 4\rho_r \Theta_{r,1}\right)\right]$$

Full system of 5 equations for $\delta, v, \Theta_{r,0}, \Theta_{r,1}, \Phi$

$$\dot\Theta_{r,0} + k\Theta_{r,1} = -\dot\Phi$$

$$\dot\Theta_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi$$

$$\dot\delta + ikv = -3\dot\Phi$$

$$\dot v + \frac{\dot a}{a}v = ik\Phi$$

$$\begin{cases} k^2 \Phi + 3\frac{\dot a}{a}\left(\dot\Phi + \frac{\dot a}{a}\Phi\right) = 4\pi G a^2 \left(\rho_{dm}\delta + 4\rho_r \Theta_{r,0}\right) \\ \text{OR} \\ k^2 \Phi = 4\pi G a^2 \left[\rho_{dm}\delta + 4\rho_r \Theta_{r,0} + \frac{3aH}{k}\left(i\rho_{dm} v + 4\rho_r \Theta_{r,1}\right)\right] \end{cases}$$

which one is more useful depends on the context

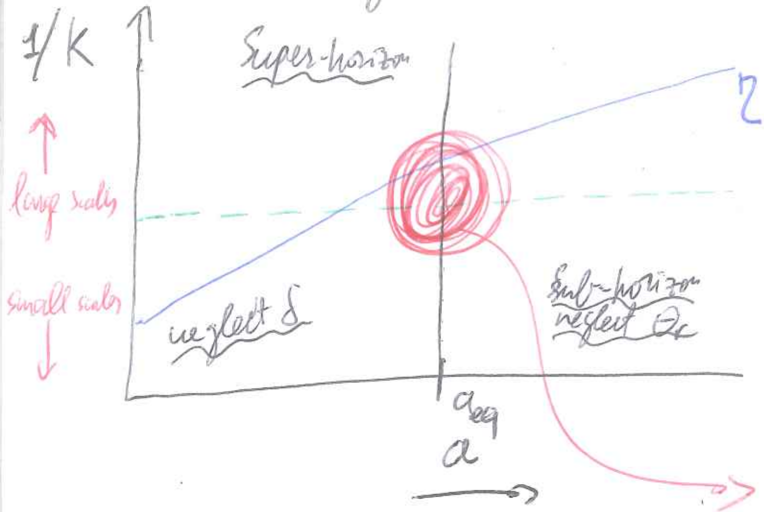
~~Analytic solutions for~~

$\rightarrow$ Very easy to solve numerically

Analytical solutions for δ are hard, exist only in certain limits (super- or sub-horizon, early- or late-time)

In general no analytic treatment for horizon crossing around equality

Solution strategy and where approximations exist



Constant Φ

a given mode has constant comoving wavenumber, enters horizon when $K\eta \sim 1$

~~Solutions~~ analytical approximations exist everywhere except here

modes crossing horizon around equality

Large scales: • early times super-horizon, drop terms proportional to k

• late times neglect radiation Θ_r, sub-horizon

• match super-horizon and sub-horizon solutions as both have $\Phi \sim$ const (crosses horizon during MD)

Small scales: • neglect matter perturbations at horizon crossing (since it is deep RD)

• late times neglect radiation, ~~Φ const~~

• match super-horizon and sub-horizon solutions

Intermediate scales which cross horizon around matter-radiation equality: no analytical solution, but transfer function is smooth, can spline large- and small-scale approximations

Now let's look at large scales first, starting super-horizon then through horizon crossing

Large scales (small K, enter horizon before equality!)

$\rightarrow k \lesssim 0.01\, h\, \mathrm{Mpc}^{-1}$

Analytical solutions for Φ through $R \rightarrow M$ transition and then horizon crossing (note crossing deep during MD)