$$\Rightarrow \frac{\ddot{a}}{a} = \frac{1}{a}\frac{d}{d\eta}(\dot{a}) \underset{\uparrow\ \dot{a}\simeq -\frac{a}{\eta}}{\simeq} \frac{1}{a}\frac{d}{d\eta}\left(-\frac{a}{\eta}\right) = -\frac{1}{a}\frac{d}{d\eta}\left(\frac{a}{\eta}\right) = -\frac{1}{a}\left(\frac{\dot{a}}{\eta} - \frac{a}{\eta^2}\right) \underset{\uparrow\ \dot{a}\simeq -\frac{a}{\eta}}{\simeq} -\frac{1}{a}\left(-\frac{a}{\eta^2} - \frac{a}{\eta^2}\right)$$

$$= -\frac{1}{a}\left(-\frac{2a}{\eta^2}\right) = \frac{2}{\eta^2}$$

$$\frac{\ddot{a}}{a} \simeq \frac{2}{\eta^2} \longrightarrow \ddot{v} + \left(k^2 - \frac{\ddot{a}}{a}\right)v = 0 \Longrightarrow \boxed{\ddot{v} + \left(k^2 - \frac{2}{\eta^2}\right)v = 0}$$

Initial conditions for $v(k,\eta)$: very early times before inflation has done most of its work

Recall $\eta = \int_{t_e}^{t} \frac{dt'}{a(t')}$ $\quad \eta_{prim} = \int_{t_e}^{*} \frac{dt'}{a(t')} < 0$ $\quad \eta_{tot} = \eta + \eta_{prim}$

At such early times $-\eta$ large, $-\eta \simeq \eta_{prim}$

$$\Rightarrow k^2 - \frac{2}{\eta^2} \simeq k^2 \longrightarrow \text{simple harmonic oscillator equation}$$

$\rightarrow$ properly normalized solution $v \propto \frac{e^{-ik\eta}}{\sqrt{2k}}$

Then with this boundary condition we can solve the full equation for v (exercise! Use $\tilde{v} = \frac{v}{\eta}$)

$$v(k,\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left[1 - \frac{i}{k\eta}\right] \xrightarrow[\substack{\text{sub-horizon}\\ \text{before inflation}\\ \text{has operated}\\ k|\eta| \gg 1}]{} \frac{e^{-ik\eta}}{\sqrt{2k}}$$

Let's check that this v is indeed solution!

Multiply by $\sqrt{2k}$, check $\frac{d^2}{d\eta^2}\left[e^{-ik\eta}\left(1-\frac{i}{k\eta}\right)\right] + \left(k^2 - \frac{2}{\eta^2}\right)e^{-ik\eta}\left(1-\frac{i}{k\eta}\right) \overset{?}{=} 0$

$$\frac{d}{d\eta}\left[e^{-ik\eta}\left(1-\frac{i}{k\eta}\right)\right] = \frac{d}{d\eta}\left[e^{-ik\eta} - \frac{i}{k\eta}e^{-ik\eta}\right] = -ike^{-ik\eta} - \frac{ke^{-ik\eta}}{k\eta} + \frac{i}{k\eta^2}e^{-ik\eta} =$$

$$= -ike^{-ik\eta} - \frac{e^{-ik\eta}}{\eta} + \frac{i}{k\eta^2}e^{-ik\eta}$$

$$\frac{d^2}{d\eta^2}\left[e^{-ik\eta}\left(1-\frac{i}{k\eta}\right)\right] = \frac{d}{d\eta}\left[-ike^{-ik\eta} - \frac{e^{-ik\eta}}{\eta} + \frac{i}{k\eta^2}e^{-ik\eta}\right] =$$

$$= -k^2e^{-ik\eta} + \frac{ik}{\eta}e^{-ik\eta} + \frac{e^{-ik\eta}}{\eta^2} + \frac{\cancel{k}}{\cancel{k}\eta^2}e^{-ik\eta} - \frac{2i}{k\eta^3}e^{-ik\eta} =$$

$$= -k^2e^{-ik\eta} + \frac{ik}{\eta}e^{-ik\eta} + \frac{e^{-ik\eta}}{\eta^2} + \frac{e^{-ik\eta}}{\eta^2} - \frac{2i}{k\eta^3}e^{-ik\eta}$$

$$\frac{d^2}{d\eta^2}\left[e^{-ik\eta}\left(1-\frac{i}{k\eta}\right)\right] + \left(\underline{k^2} - \underline{\frac{2}{\eta^2}}\right)e^{-ik\eta}\left(1-\frac{i}{k\eta}\right) =$$

$$= \cancel{-k^2e^{-ik\eta}} + \cancel{\frac{ik}{\eta}e^{-ik\eta}} + \cancel{\frac{e^{-ik\eta}}{\eta^2}} + \cancel{\frac{e^{-ik\eta}}{\eta^2}} - \cancel{\frac{2i}{k\eta^3}e^{-ik\eta}} + \cancel{k^2e^{-ik\eta}} - \cancel{\frac{ik}{\eta}e^{-ik\eta}}$$

$$\cancel{-\frac{2}{\eta^2}e^{-ik\eta}} + \cancel{\frac{2i}{k\eta^3}e^{-ik\eta}} = 0! \quad ✓$$

So the solution to the Mukhanov-Sasaki equation is indeed:

$$v = \frac{e^{-ik\eta}}{\sqrt{2k}}\left[1 - \frac{i}{k\eta}\right]$$

After inflation has done its job $k|\eta| \ll 1$

↳ mode has exited the horizon

Variance of the super-horizon GW amplitude: $\lim_{-k\eta\to 0} |v(k,\eta)|^2 \frac{16\pi G}{a^2}$

$$\lim_{-k\eta\to 0} v(k,\eta) = -\frac{e^{-ik\eta}}{\sqrt{2k}}\frac{i}{k\eta} = -\frac{ie^{-ik\eta}}{\sqrt{2k^3\eta^2}}$$

Recall then that $\hat{\tilde{h}} = v\hat{a}^{\dagger} + v^{*}\hat{a}$ $\quad \tilde{h} \equiv \frac{ah}{\sqrt{16\pi G}}$

$\Rightarrow h \propto \frac{v}{a}$ How does h evolve?

- Early times: (sub-horizon) $v \propto \frac{e^{-ik\eta}}{\sqrt{2k}}$ $h \propto \frac{v}{a} \propto \frac{1}{a}$ inflation reduces amplitude of mode

Late times: (after horizon exit) $v \propto \frac{e^{-ik\eta}}{k^{3/2}\eta}$ $h \propto \frac{v}{a} \propto \frac{1}{\eta a} \propto$ Const since $\eta \simeq -\frac{1}{aH}$

So after the mode exits the horizon it is roughly constant and so is the power spectrum (constant in time, not in k!)

this constant then sets initial conditions for GWs generated by inflation

- $$P_h(k) = \frac{16\pi G}{a^2} \lim_{-k\eta \to 0} |v(k,\eta)|^2 =$$

$$= \frac{16\pi G}{a^2} \frac{1}{2k^3\eta^2} = \frac{8\pi G}{k^3 a^2 \eta^2} \simeq \frac{8\pi G H^2}{k^3}$$

$\eta \simeq -\frac{1}{aH}$, $a\eta \simeq -\frac{1}{H}$, $\frac{1}{a^2\eta^2} \simeq H^2$

$$\boxed{P_h(k) \simeq \left. \frac{8\pi G H^2}{k^3} \right|_{\text{horizon exit : } k = aH}}$$

Scale-invariant spectrum since dimensionless power spectrum

$\Delta \propto k^3 P(k) \propto$ Const

Detection of inflationary GWs would measure $H^2 \propto \rho \propto V$ during inflation

Typically $V \gtrsim 10^{15}$ GeV, well beyond what can be probed terrestrially

Note $H^2 \propto \frac{\rho}{M_{Pl}^2} \rightarrow P(k) \propto \frac{\rho^{\cancel{2}}}{\boxed{M_{Pl}^4}} \rightarrow$ Huge suppression!

- Final remarks:

- Fluctuations in h are Gaussian as for simple harmonic oscillator
- $P_{h_+} = P_{h_\times} \rightarrow$ total power spectrum $= 2 \times P_h(k)$!

V Inflationary scalar perturbations

Previously we found the power spectrum of h emerging from inflation
Now we want to find the power spectrum of $\Psi = -\Phi \Rightarrow P_\Psi = P_\Phi$
Then using the relations found earlier we can relate this to the spectra of other variables

$$\Phi(k,\eta_i) = 2\Theta_0(k,\eta_i) = 2\mathcal{N}_0(k,\eta_i)$$

$$\delta(k,\eta_i) = \delta_b(k,\eta_i) = 3\Theta_0(k,\eta_i)$$

$$\Theta_1(k,\eta_i) = \mathcal{N}_1(k,\eta_i) = \frac{i v_b(k,\eta_i)}{3} = \frac{i v(k,\eta_i)}{3} = -\frac{k\Phi(k,\eta_i)}{6aH}$$

This is <u>difficult</u> because perturbations to the inflaton ϕ are coupled to Ψ. So we will start by ignoring this coupling

Strategies:

a) perturbations to ϕ ignoring Ψ
(Ψ negligible until ~~ϕ moves~~ mode moves outside the horizon)

$\downarrow$

Linear combination of $\delta\phi$ and Ψ conserved on super-horizon scales

$\downarrow$

Convert $\delta\phi$ spectrum to Ψ spectrum

b) Work in spatially flat slicing: gauge where g_{ij} is unperturbed

$\downarrow$

find gauge-invariant variable proportional to $\delta\phi$ in spatially flat slicing

$\downarrow$

Convert back to conformal Newtonian gauge

Scalar field perturbations around a smooth background

Start by looking at spectrum for $\delta\phi$ neglecting Ψ

$$\phi(\bar{x},t) = \underbrace{\phi^{(0)}(t)}_{\text{zero-order homogeneous}} + \underbrace{\delta\phi(\bar{x},t)}_{\text{first-order perturbation}}$$

Unperturbed metric $g_{00} = -1$ $g_{ij} = \delta_{ij} a^2$ (no Ψ, Φ)

Conservation of energy-momentum tensor

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu{}_{\alpha\mu} T^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu} T^\mu{}_\alpha = 0$$

Æ $\hookrightarrow$ ${}^0{}_0$ component expanded to 1st order: equation for $\delta\phi$

Recall $\Gamma^0{}_{ij} = \delta_{ij} \dot{a} a = \delta_{ij} a^2 H$ $\Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \delta_{ij} \frac{\dot{a}}{a} = \delta_{ij} H$

All other Γs $= 0$

All Γs are 0 or 0th order, so ~~perturbations~~ 1st order pieces only come from perturbations to $T^\mu{}_\nu$ $[\delta T^\mu{}_\nu]$

$\nu = 0$ component of perturbed conservation equation

$$\nabla_\mu \delta T^\mu{}_0 = \partial_\mu \delta T^\mu{}_0 + \Gamma^\mu{}_{\alpha\mu} \delta T^\alpha{}_0 - \Gamma^\alpha{}_{0\mu} \delta T^\mu{}_\alpha =$$

$$\cancel{= \partial_0 \delta T^0{}_0 + \partial_i \delta T^i{}_0 + \Gamma^i{}_{0i} \delta T^0{}_0 - \Gamma}$$

$$= \frac{\partial \delta T^0{}_0}{\partial t} + \underbrace{i k_i \delta T^i{}_0}_{\text{Fourier space}} + \underbrace{3H}_{\delta_{ii}=3} \delta T^0{}_0 - H \delta T^i{}_i$$

$$\Rightarrow \frac{\partial \delta T^0{}_0}{\partial t} + i k_i \delta T^i{}_0 + 3H \delta T^0{}_0 - H \delta T^i{}_i = 0$$

So we need the perturbed pieces of $T^{\mu}{}_{\nu}$

$$T^{\alpha}{}_{\beta} = g^{\alpha\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^\beta} - g^{\alpha}{}_{\beta}\left[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} + V(\phi)\right]$$

$$\delta T^{i}{}_{0} = g^{i\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^0} - \cancelto{0}{g^{i}{}_{0}}\left[\dots\right] = g^{i\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^0} =$$

$$= 0 \quad \text{since} \quad \nu = i \quad (g^{i\nu} = a^{-2}\delta_{i\nu}) \quad \text{and} \quad \frac{\partial\phi^{(0)}}{\partial x^i} = 0 \Rightarrow T^{(0)\,i}{}_{0} = 0$$

0th order

let's look at 1st order piece : set $\frac{\partial\phi}{\partial x^i} = \frac{\partial\delta\phi}{\partial x^i} = ik_i\delta\phi$

~~$\delta T^{i}{}_{0} = g^{i\nu}\frac{\partial\delta\phi}{\partial x^\nu}\frac{\partial\delta\phi}{\partial x^0} = g^{ii}\frac{\partial\delta\phi}{\partial x^i}\frac{\partial\delta\phi}{\partial x^0} = a^{-2}ik_i\delta\phi\frac{\partial\delta\phi}{\partial t} = ik_i\frac{\partial\delta\phi^{(0)}}{a^2}$~~

~~$\frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta} = \frac{\dot{}}{a}$~~

since $\frac{\partial\phi^{(0)}}{\partial x^i} = 0$ must set $\frac{\partial\phi}{\partial x^i} = \frac{\partial\delta\phi}{\partial x^i}$

$$\delta T^{i}{}_{0} = \left[g^{i\nu}\frac{\partial\phi}{\partial x^\nu}\frac{\partial\phi}{\partial x^0}\right]\Bigg|_{\text{1st order}} \simeq g^{i\nu}\frac{\partial\delta\phi}{\partial x^\nu}\frac{\partial\phi^{(0)}}{\partial x^0} = g^{ii}\frac{\partial\delta\phi}{\partial x^i}\frac{\partial\phi^{(0)}}{\partial t} =$$

$$= \cancel{a^{-2}ik_i\delta\phi} = \underbrace{a^{-2}ik_i\delta\phi}_{\text{Fourier space}}\frac{\partial\phi^{(0)}}{\partial t} = \frac{ik_i}{a^3}\frac{\partial\phi^{(0)}}{\partial\eta}\delta\phi$$

$\frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta} = \frac{\dot{}}{a}$

$$\delta T^{i}{}_{0} = \frac{ik_i}{a^3}\frac{\partial\phi^{(0)}}{\partial\eta}\delta\phi = \frac{ik_i}{a^3}\dot{\phi}^{(0)}\delta\phi$$

$$T^{0}{}_{0} = g^{00}\left(\frac{\partial\phi}{\partial x^0}\right)^2 - \frac{1}{2}g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} - V(\phi) = \frac{1}{2}g^{00}\left(\frac{\partial\phi}{\partial x^0}\right)^2 - \frac{1}{2}g^{ii}\left(\frac{\partial\phi}{\partial x^i}\right)^2 - V(\phi) -$$

$\phi = \phi^{(0)} + \delta\phi$

$g^{00} = -1$

$g^{ii} = a^{-2}$

$$= -\frac{1}{2}\left(\frac{\partial\phi^{(0)}}{\partial t} + \frac{\partial\delta\phi}{\partial t}\right)^2 - \frac{1}{2a^2}\cancelto{\text{2nd order}}{\frac{\partial\delta\phi}{\partial x^i}\frac{\partial\delta\phi}{\partial x^i}} - V(\phi^{(0)} + \delta\phi) \simeq$$

$$\simeq -\frac{1}{2}\left[\underbrace{\left(\frac{\partial \phi^{(0)}}{\partial t}\right)^2}_{\text{0th order}} + 2\frac{\partial \phi^{(0)}}{\partial t}\frac{\partial \delta\phi}{\partial t} + \overset{\text{2nd order}}{\cancel{\left(\frac{\partial \delta\phi}{\partial t}\right)^2}}\right] - \underbrace{\underbrace{V(\phi^{(0)})}_{\text{0th order}} - \delta\phi\frac{\partial V(\phi)}{\partial \phi}}_{\text{Taylor expansion}}$$

$$\Rightarrow \delta T^{0}{}_{0}\Big|_{\text{1st order}} \simeq -\frac{\partial \phi^{(0)}}{\partial t}\frac{\partial \delta\phi}{\partial t} - V'\delta\phi \underset{\frac{d}{dt}=\frac{\cdot}{a}}{=} -\frac{\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi$$

$$\delta T^{0}{}_{0} = \frac{-\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi$$

Exercise: $\delta T^{i}{}_{j}$

Summary

$$\delta T^{i}{}_{0} \simeq \frac{ik_i}{a^3}\phi^{(0)}\delta\phi \qquad \delta T^{0}{}_{0} \simeq \frac{-\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi$$

$$\delta T^{i}{}_{j} = \delta_{ij}\left(\frac{\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi\right)$$

Back to conservation equation

$$\frac{\partial \delta T^{0}{}_{0}}{\partial t} + ik_i\delta T^{i}{}_{0} + 3H\delta T^{0}{}_{0} - H\delta T^{i}{}_{i} = \left(\frac{\partial}{\partial t} + 3H\right)\delta T^{0}{}_{0} + ik_i\delta T^{i}{}_{0} - H\delta T^{i}{}_{i} =$$

$$= \left(\frac{1}{a}\frac{\partial}{\partial \eta} + 3H\right)\delta T^{0}{}_{0} + ik_i\delta T^{i}{}_{0} - H\delta T^{i}{}_{i} = 0$$

Putting everything together

$$\left(\frac{1}{a}\frac{\partial}{\partial \eta} + 3H\right)\left(\frac{-\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi\right) - \frac{k^2}{a^3}\dot\phi^{(0)}\delta\phi - \underbrace{3}_{\delta_{ii}}H\left(\frac{\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - V'\delta\phi\right) = 0$$

$$\hookrightarrow = \frac{1}{a}\frac{\partial}{\partial \eta}\left[\frac{-\dot\phi^{(0)}\dot{\delta\phi}}{a^2}\right] - \frac{1}{a}\frac{\partial}{\partial \eta}(V'\delta\phi) - 3H\frac{\dot\phi^{(0)}\dot{\delta\phi}}{a^2} - \cancel{3HV'\delta\phi} - \frac{k^2}{a^3}\dot\phi^{(0)}\delta\phi$$

$$- 3H\frac{\dot\phi^{(0)}\dot{\delta\phi}}{a^2} + \cancel{3HV'\delta\phi}\,.$$

$$= \underbrace{\frac{1}{a}\frac{\partial}{\partial\eta}\left[\frac{-\dot\phi^{(0)}\delta\dot\phi}{a^2}\right]} - \underbrace{\frac{1}{a}\frac{\partial}{\partial\eta}(V'\delta\phi)} - \frac{6H\dot\phi^{(0)}\delta\dot\phi}{a^2} - \frac{k^2}{a^3}\dot\phi^{(0)}\delta\phi = *$$

$$\frac{1}{a}\frac{\partial}{\partial\eta}\left[\frac{-\dot\phi^{(0)}\delta\dot\phi}{a^2}\right] = -\frac{\ddot\phi^{(0)}\delta\dot\phi}{a^3} - \frac{\dot\phi^{(0)}\delta\ddot\phi}{a^3} + \frac{2\dot\phi^{(0)}\delta\dot\phi}{a^4}\dot a =$$

$\left(\frac{\dot a}{a^4} = \frac{1}{a^2}\frac{\dot a}{a^2} = \frac{H}{a^2}\right.$ since $H = \frac{1}{a}\frac{da}{dt} = \frac{\dot a}{a^2}$

$$= -\frac{\ddot\phi^{(0)}\delta\dot\phi}{a^3} - \frac{\dot\phi^{(0)}\delta\ddot\phi}{a^3} + \frac{2H\dot\phi^{(0)}\delta\dot\phi}{a^2}$$

$$-\frac{1}{a}\frac{\partial}{\partial\eta}(V'\delta\phi) = -\frac{1}{a}\left[\frac{\partial V'}{\partial\eta}\delta\phi + V'\delta\dot\phi\right] = -\frac{1}{a}\left[\underbrace{\frac{\partial V'}{\partial\phi^{(0)}}}_{V''}\underbrace{\frac{\partial\phi^{(0)}}{\partial\eta}}_{\dot\phi^{(0)}}\delta\phi + V'\delta\dot\phi\right] =$$

$$= -\frac{1}{a}\left(V''\dot\phi^{(0)}\delta\phi + V'\delta\dot\phi\right) = -\frac{1}{a}V''\dot\phi^{(0)}\delta\phi - \frac{1}{a}V'\delta\dot\phi$$

$$* = -\frac{\ddot\phi^{(0)}\delta\dot\phi}{a^3} - \frac{\dot\phi^{(0)}\delta\ddot\phi}{a^3} + \frac{2H\dot\phi^{(0)}\delta\dot\phi}{a^2} - \frac{V''\dot\phi^{(0)}\delta\phi}{a} - \frac{V'\delta\dot\phi}{a}$$

$$- \frac{6H\dot\phi^{(0)}\delta\dot\phi}{a^2} - \frac{k^2}{a^3}\dot\phi^{(0)}\delta\phi = 0$$

$$\Rightarrow -\dot\phi^{(0)}\delta\ddot\phi + \delta\dot\phi\underbrace{\left(-\ddot\phi^{(0)} - 4aH\dot\phi^{(0)} - a^2V'\right)}_{\text{scalar field equation of motion}} + \delta\phi\left(\cancel{-a^2\underbrace{V''}_{\text{Small, }\propto\epsilon,\delta}} - k^2\dot\phi^{(0)}\right) = 0$$

$\times a^3$

$$\ddot\phi^{(0)} + 2aH\dot\phi^{(0)} + a^2V' = 0 \Rightarrow -\ddot\phi^{(0)} - 4aH\dot\phi^{(0)} - a^2V' = -2aH\dot\phi^{(0)}$$

$$\Rightarrow \cancel{-\dot\phi^{(0)}}\,\delta\ddot\phi - 2aH\cancel{\dot\phi^{(0)}}\,\delta\dot\phi - k^2\cancel{\dot\phi^{(0)}}\,\delta\phi = 0$$

$$\Rightarrow \boxed{\delta\ddot\phi + 2aH\delta\dot\phi + k^2\delta\phi = 0}$$

Equation for perturbations to inflaton scalar field in unperturbed $g_{\mu\nu}$

Now recall what was the evolution equation for tensor perturbations

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} + K^2 h = \ddot{h} + 2aH\dot{h} + K^2 h = 0$$

Identical in form to $\delta\phi$ equation! So for $\delta\phi$ we can borrow the results from tensor perturbations, minus the factor of $16\pi G$ which we don't need here

$$P_h(k) = \frac{16\pi G}{a^2}\frac{1}{2k^3\eta^2} \underset{\overset{\uparrow}{\eta \simeq -\frac{1}{aH}}}{\simeq} \frac{8\pi G H^2}{K^3} \underset{\times\frac{1}{16\pi G}}{\Longrightarrow} \boxed{P_{\delta\phi}(k) \simeq \frac{H^2}{2k^3}}$$

So also the power spectrum of scalar field fluctuations from inflation is (nearly...) scale-invariant, $\Delta \propto k^3 P(k) \propto \text{const}$

Super-horizon perturbations

So far we neglected Ψ, which is valid when the perturbation is sub-horizon ($k \gg aH$), but when a mode exits the horizon Ψ becomes important and we can find a linear combination of $\delta\phi$ and Ψ which is conserved.

Symbolically, evolution of inflationary perturbations:

$$(\text{mostly})\ \delta\phi \longrightarrow \Psi \text{"+"} \delta\phi \;/\; \Psi \text{"+"} \delta T^{\mu}{}_{\nu}$$

Goal: find linear combination of Ψ and $\delta\phi$ conserved ~~at horizon~~ ~~crossing, then~~ on sub-horizon scales, will depend on $\delta\phi$ at horizon crossing, evaluate it after inflation in terms of $\underbrace{\Psi}_{\text{after inflation}} (\propto \underbrace{\delta\phi}_{\text{before inflation}})$, finally

relate P_Ψ to $P_{\delta\phi}$

$$\hookrightarrow = \frac{H^2}{2k^3}$$

Start again from covariant conservation of energy-momentum:

$$\underbrace{\frac{\partial \delta T^0{}_0}{\partial t} + ik_i\delta T^i{}_0 + 3H\delta T^0{}_0 - H\delta T^i{}_i}_{\text{saw earlier}} = -3(P+\rho)\frac{\partial \Psi}{\partial t}$$

Why were we justified in neglecting Ψ previously? If $(P+\rho)\frac{\partial\Psi}{\partial t} \ll$ RHS, this was fine. Example: 1st term

$$\frac{\partial \delta T^0{}_0}{\partial t} \gg (P+\rho)\frac{\partial \Psi}{\partial t} \implies \Psi \ll \frac{\delta T^0{}_0}{P+\rho}$$

Is this true? Look at perturbed Einstein equations we found

$$k^2\Phi + 3\frac{\dot a}{a}\left(\dot\Phi - \Psi\frac{\dot a}{a}\right) = -4\pi G a^2\delta T^0{}_0 \xrightarrow[\frac{\dot a}{a}=aH]{\Phi=-\Psi} \boxed{k^2\Psi + 3aH(\dot\Psi + aH\Psi) = 4\pi G a^2\delta T^0{}_0}\,{}^{*}$$

$*$ LHS $\sim k^2\Psi \sim a^2H^2\Psi$ at horizon crossing

$$\Rightarrow \cancel{a^2}H^2\Psi \sim G\cancel{a^2}\delta T^0{}_0 \implies \Psi \sim \frac{G\delta T^0{}_0}{H^2} \sim \frac{\delta T^0{}_0}{\rho} \qquad \left[H^2 \sim G\rho \Rightarrow \frac{G}{H^2} \sim \frac{1}{\rho}\right]$$

$$= \frac{P+\rho}{\rho}\left(\frac{\delta T^0{}_0}{P+\rho}\right)$$

For $\Psi \ll \frac{\delta T^0{}_0}{P+\rho}$ we require $\frac{P+\rho}{\rho} \ll 1$ which is the case during inflation since $P \simeq -\rho$ $\left[\text{in fact } \frac{P+\rho}{\rho} = \frac{2\varepsilon}{3} \ll 1\right]$

So for slow-roll models of inflation we can safely neglect Ψ when computing perturbations $\delta\phi$ FOR SUBHORIZON MODES

However at some point $\underbrace{\frac{P+\rho}{\rho}}_{=1+w} \ll 1$ won't hold anymore

$\nearrow$ ~ 0 during inflation $w \simeq -1$

$\searrow$ $\frac{4}{3}$ during RD

E.g. radiation domination

$$\delta T^0{}_0 = -4\rho_r\Theta_0 \qquad \rho+p = 4\frac{\rho_r}{3} \Rightarrow \frac{\delta T^0{}_0}{\rho+p} \sim -3\Theta_0$$

But we know that

$$-\Phi(k,\eta_i) = \Psi(k,\eta_i) = -2\Theta_0(k,\eta_i)$$

↳ η_i early on but after inflation has operated!

Not true that $\Psi = -2\Theta_0 \ll \frac{\delta T^0{}_0}{\rho+p} \sim -3\Theta_0$!!!

Before the end of inflation Ψ has to grow in importance relative to $\delta T^0{}_0$: perturbations to the metric need to become comparable in importance to those in the energy-momentum tensor!

Define new variable

$$\zeta \equiv -\frac{ik_i\,\delta T^0{}_i H}{k^2(\rho+P)} - \Psi$$

For sub-horizon modes and modes which just crossed the horizon

$$\zeta_{sub} = \frac{-ik_i\,\delta T^0{}_i H}{k^2(\rho+P)} \overset{\text{negligible}}{\cancel{-\Psi}} \underset{\Psi\to 0}{\simeq} \frac{-i\cancel{k_i}}{\cancel{k^2}} H \times -\frac{ik_i}{a^3}\dot\phi^{(0)}\delta\phi\left(\frac{a}{\dot\phi^{(0)}}\right)^2 = -\frac{aH\delta\phi}{\dot\phi^{(0)}}$$

$$P+\rho = \left(\frac{d\phi^{(0)}}{dt}\right)^2 = \frac{1}{a^2}\left(\dot\phi^{(0)}\right)^2$$

$$-\delta T^i{}_0 = +\delta T^0{}_i = -\frac{ik_i}{a^3}\dot\phi^{(0)}\delta\phi$$

After inflation ends (exercise!)

$$ik_i\,\delta T^0{}_i = 4ak\rho_r\Theta_1 \longrightarrow \delta T^0{}_i = -i\frac{4ak\rho_r\Theta_1}{k_i}$$

$$\zeta_{super} = \frac{-ik_i\,\delta T^0{}_i H}{k^2(\rho+P)} - \Psi \simeq -\frac{i\cancel{k_i}}{k^2} \times \frac{-i4ak\rho_r\Theta_1}{\cancel{k_i}}\frac{H}{(\rho_r+P_r)} - \Psi$$

$$= -\frac{4aH}{k}\Theta_1\frac{\rho_r}{(\rho_r+P_r)} - \Psi \underset{\uparrow}{=} -\frac{3aH\Theta_1}{k} - \Psi \underset{\uparrow}{=} -\frac{3a\cancel{H}}{\cancel{k}}\frac{k\Psi}{6a\cancel{H}} - \Psi = -\frac{1}{2}\Psi - \Psi = -\frac{3}{2}\Psi$$

$$\frac{\rho_r+P_r}{\rho_r} = 1+w_r = \frac{4}{3}$$

$$\Theta_1(k,\eta_i) = -\frac{k\Phi(k,\eta_i)}{6aH} = \frac{k\Psi}{6aH}(k,\eta_i)$$