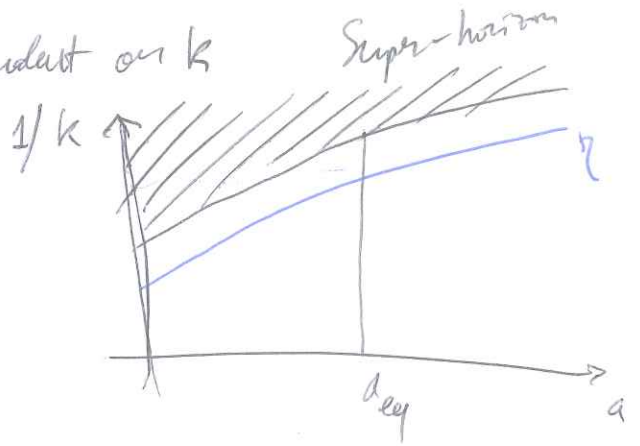


Super-horizon solution

- $k\eta \ll 1 \rightarrow$ Neglect all terms dependent on k

Choose Einstein equation with derivations
(else $\sim \frac{1}{k}$ terms)



$$\begin{cases} \dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi} \\ \dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi \\ \dot{\delta} + ikv = -3\dot{\Phi} \\ \dot{v} + \frac{\dot{a}}{a}v = ik\Phi \\ k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 4\pi G a^2 (\rho_{dm}\delta + 4p_r\Theta_{r,0}) \end{cases} \rightarrow \begin{cases} \dot{\Theta}_{r,1} = 0 \\ \dot{v} + \frac{\dot{a}}{a}v = 0 \end{cases}$$

decouple

We go from 5 to 3 effective variables

$$\Rightarrow \begin{cases} \dot{\Theta}_{r,0} \approx -\dot{\Phi} \\ \dot{\delta} \approx -3\dot{\Phi} \\ 3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) \approx 4\pi G a^2 (\rho_{dm}\delta + 4p_r\Theta_{r,0}) \end{cases} \rightarrow \delta - 3\dot{\Theta}_{r,0} = 0 \Rightarrow \delta - 3\Theta_{r,0} = \text{const}$$

- $\delta - 3\Theta_{r,0} = \text{const}$ but initial conditions tell us $\delta(k, \eta_i) = 3\Theta_{r,0}(k, \eta_i)$

$$\rightarrow \text{const} = 0$$

$$\text{so we can set } \delta = 3\Theta_{r,0} \rightarrow \Theta_{r,0} = \frac{\delta}{3}$$

$$3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 4\pi G a^2 (\rho_{dm}\delta + 4p_r\Theta_{r,0})$$

$$\Downarrow \text{Define } y \equiv \frac{a}{a_{eq}} \Rightarrow \rho_{dm} a^{-3} = \rho_{r,0} a_{eq}^{-4} \Rightarrow a_{eq} = \frac{\rho_{r,0}}{\rho_{dm}}$$

$$\frac{p_r}{\rho_r} = \frac{p_{r,0} a^4}{\rho_{r,0} a^3} = \frac{p_{r,0}}{\rho_{r,0}} \frac{1}{a} = \frac{a_{eq}}{a}$$

$$\begin{aligned} 3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) &= 4\pi G a^2 (\rho_{dm}\delta + \frac{4}{3}p_r\delta) = 4\pi G a^2 \rho_{dm}\delta \left(1 + \frac{4}{3}\frac{p_r}{\rho_{dm}}\right) = 4\pi G a^2 \rho_{dm}\delta \left(1 + \frac{4}{3}\frac{a_{eq}}{a}\right) \\ &= 4\pi G a^2 \rho_{dm}\delta \left(1 + \frac{4}{3y}\right) \end{aligned}$$

$$\Rightarrow \text{Einstein equation becomes } 3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 4\pi G a^2 \rho_{dm}\delta \left(1 + \frac{4}{3y}\right)$$

$$3 \frac{\dot{a}}{a} \left(\dot{\Phi} + \frac{\dot{a}}{a} \Phi \right) = 4\pi G a^2 \rho_{dm} \delta \left(1 + \frac{4}{3y} \right)$$

y new evolution variable, replaces τ, a (ignoring baryons!)

So since $\delta = 3\Theta_{r,0}$ we have reduced the problem to equations

for δ and $\Phi \rightarrow$ turn 2 1st order equations $(\delta, \dot{\Phi})$ into 1 2nd order equation $(\ddot{\Phi})$

Change variable to y

$$\frac{d}{d\tau} = \frac{dy}{d\tau} \frac{d}{dy} = \frac{d}{d\tau} \left(\frac{a}{a_{eq}} \right) \frac{d}{dy} = \frac{1}{a_{eq}} \left(\frac{da}{d\tau} \right) \frac{d}{dy} = \frac{\dot{a}}{a_{eq}} \frac{d}{dy} =$$

$$\dot{a} = \frac{da}{d\tau} = a \frac{da}{dt} = a^2 \left(\frac{1}{a} \frac{da}{dt} \right) = a^2 H$$

$$= \frac{a^2 H}{a_{eq}} \frac{d}{dy} = \left(\frac{a}{a_{eq}} \right) a H \frac{d}{dy} = a H y \frac{d}{dy} = a H y' \quad \left[' \equiv \frac{d}{dy} \right]$$

$$\rightarrow a H y'$$

Einstein equation becomes

$$3 \frac{\dot{a}}{a} \left(\dot{\Phi} + \frac{\dot{a}}{a} \Phi \right) = 4\pi G a^2 \rho_{dm} \delta \left(1 + \frac{4}{3y} \right)$$

$$\downarrow \frac{\dot{a}}{a} = a \left(\frac{1}{a} \frac{da}{dt} \right) = aH$$

$$3aH (aHy' \Phi' + aH \Phi)$$

$$\downarrow$$

$$\cancel{3a^2 H^2 y} \Phi' + \cancel{3a^2 H^2} \Phi$$

$$\downarrow y = \frac{a_{dm}}{r}, \quad y+1 = \frac{a_{dm} + a_{r}}{r} = \frac{\rho}{\rho}$$

$$\Rightarrow \frac{y}{y+1} = \frac{a_{dm} \rho}{r \rho} \rightarrow a_{dm} = \frac{y}{y+1} r$$

$$= 4\pi G \rho a^2 \frac{y}{y+1} \delta \left(1 + \frac{4}{3y} \right)$$

$$\downarrow \frac{8\pi G}{3} \rho = H^2 \rightarrow 4\pi G \rho = \frac{2}{3} H^2$$

$$= \frac{2}{3} a^2 H^2 \frac{y}{y+1} \delta \left(1 + \frac{4}{3y} \right)$$

$$\Downarrow$$

$$y\Phi' + \Phi = \frac{y}{2(y+1)} \delta \left(1 + \frac{4}{3y} \right) = \delta \frac{y + \frac{4}{3}}{2(y+1)} = \delta \frac{3y+4}{6(y+1)}$$

We want to turn 2 1st order equations into 1 2nd order equation

$$\begin{cases} \delta' = -3\Phi' \\ y\Phi' + \Phi = \frac{3y+4}{6(y+1)} \delta \end{cases}$$

$$\delta' = \left[\frac{6(y+1)}{3y+4} (y\Phi' + \Phi) \right]' = -3\Phi'$$

... a mess!

$$\frac{d}{dy} \left[\frac{6(y+1)}{3y+4} (y\Phi' + \Phi) \right] = \frac{6\Phi}{(3y+4)^2} + \frac{48+60y+36y^2}{(3y+4)^2} \Phi' + \frac{y(y+1)}{(3y+4)} \Phi'' = -3\Phi'$$

straightforward but tedious exercise!

(I use Mathematica)

(algebra)

$$\frac{6}{(3y+4)^2} \Phi + \frac{3(32+56y+24y^2)}{(3y+4)^2} \Phi' + \frac{6y(y+1)(3y+4)}{(3y+4)^2} \Phi'' = 0$$

$$\Rightarrow \Phi'' + \frac{24y^2 + 56y + 32}{2y(y+1)(3y+4)} \Phi' + \frac{\Phi}{y(y+1)(3y+4)} = 0$$

change of variable $u \equiv \frac{y^3}{\sqrt{1+y}} \Phi$ $u' = y^2 \frac{[(6+5y)\Phi + 2y(1+y)\Phi']}{2(1+y)^{3/2}}$

$$u'' + u' \left[-\frac{2}{y} + \frac{3/2}{1+y} - \frac{3}{3y+4} \right] = 0$$

Note: no u term! Just u'' and u' , so it's a "fake" 2nd order equation, really just 1st order equation for u'
Integrable!

$$u'' = \frac{d^2 u}{dy^2} = \frac{du}{dy} \left[\frac{2}{y} - \frac{3/2}{1+y} + \frac{3}{3y+4} \right]$$

$$\rightarrow \frac{du'}{dy} = u' \left[\frac{2}{y} - \frac{3/2}{1+y} + \frac{3}{3y+4} \right] \Rightarrow$$

$$\rightarrow \int \frac{du'}{u'} = \int dy \left[\frac{2}{y} - \frac{3/2}{1+y} + \frac{3}{3y+4} \right]$$

$$\rightarrow \ln(u') = \text{const} + 2\ln(y) - \frac{3}{2}\ln(1+y) + \ln(3y+4)$$

$$\rightarrow \frac{du'}{dy} = A \frac{y^2 (3y+4)}{(1+y)^{3/2}} \quad \text{Recall} \quad u = \frac{y^3}{\sqrt{1+y}} \Phi$$

$$u(y) = \frac{y^3}{\sqrt{1+y}} \Phi = A \int_0^y \frac{y'^2 (3y'+4)}{(1+y')^{3/2}} dy' + \text{const}$$

small y ($y \rightarrow 0$)

$$y^3 \Phi \approx A \int_0^y dy' 4y'^2 = \frac{4A}{3} y^3 \Rightarrow \frac{4A}{3} \Phi$$

$$\lim_{y \rightarrow 0} \Phi = \frac{4A}{3} \Rightarrow A = \lim_{y \rightarrow 0} \frac{3}{4} \Phi = \frac{3\Phi(0)}{4}$$

$$\Rightarrow \Phi = \frac{3\Phi(0)}{4} \left(\frac{y^3}{\sqrt{1+y}} \right)^{-1} \int_0^y \frac{y'^2 (3y'+4)}{(1+y')^{3/2}} dy' = \frac{3\Phi(0)}{4} \frac{y^3}{\sqrt{1+y}}$$

I use
Mathematica

Exercise!

Hint: substitution $x \equiv \sqrt{1+y}$

$$\frac{2}{15} \left(16 + \frac{-16+y(-8+y(2+9y))}{\sqrt{1+y}} \right)$$

$$= \frac{\Phi(0)}{10} \frac{\sqrt{1+y}}{y^3} \frac{1}{15} \left(16 + \frac{y[y(2+9y) - 8] - 16}{\sqrt{1+y}} \right) =$$

$$= \frac{\Phi(0)}{10} \frac{1}{y^3} (16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16)$$

~~Final Answer~~

$$\boxed{\bar{\Phi} = \frac{\Phi(0)}{10} \frac{1}{y^3} (16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16)}$$

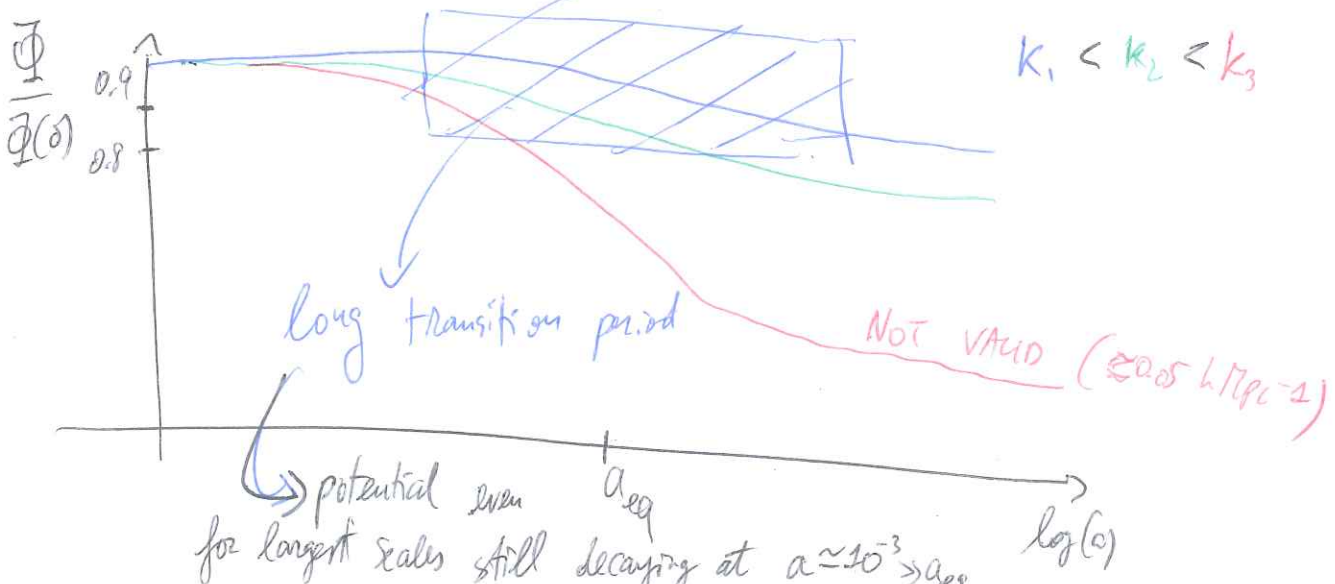
Gravitational potential $\left\{ \begin{array}{l} \text{for large-scale modes} \\ \text{on super-horizon scales (valid even during RD)} \\ y = \frac{q}{a_{eq}} \propto a \\ \text{ignoring baryons} \end{array} \right.$

Properties:

- $y \rightarrow 0$ ($a \rightarrow 0$, early times) $\Rightarrow \bar{\Phi} \approx \Phi(0)$ as expected!
- $y \gg 1$ ($a \gg a_{eq}$, deep RD) $\Rightarrow \bar{\Phi} \approx \frac{\Phi(0)}{10} \frac{1}{y^3} 9y^3 = \frac{9}{10} \Phi(0)$

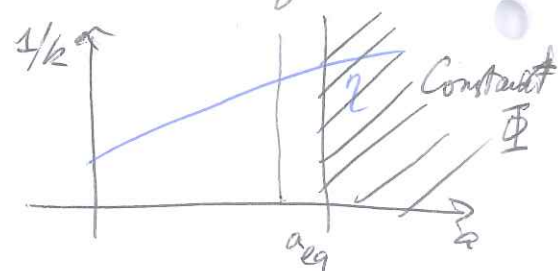
So even if the mode is super-horizon (no causal physics operates) the potential still drops by 10% when going through matter-radiation equality $(z \approx 0.01 \text{ hMpc}^{-1})$ (the RD \rightarrow MD transition is long)

Solution works quite well down to $k \lesssim 0.01 \text{ hMpc}^{-1}$,



Horizon crossing

We will prove that when the mode crosses the horizon during MD, Φ remains constant (as long as MD holds)



This time we cannot neglect k terms, but we can neglect radiation. Full set of equations (use no-derivatives Einstein equation.)

$$\begin{cases} \dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi} & \text{neglect radiation} \\ \dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi & \text{neglect radiation} \end{cases}$$

$$\begin{cases} \delta + ikv = -3\dot{\Phi} \\ \dot{v} + \frac{\dot{a}}{a}v = ik\Phi \end{cases}$$

$$k^2\Phi = 4\pi G a^2 \left[\rho_{dm}\delta + \frac{3aH}{k} (i\rho_{dm}v + \dots) \right] \approx 4\pi G a^2 \rho \left[\delta + \frac{3aHiv}{k} \right] = \frac{3}{2} a^2 H^2 \left[\delta + \frac{3aHiv}{k} \right]$$

$$\Rightarrow \begin{cases} \delta + ikv = -3\dot{\Phi} \\ \dot{v} + \frac{\dot{a}}{a}v = ik\Phi \\ k^2\Phi = \frac{3}{2} a^2 H^2 \left(\delta + \frac{3aHiv}{k} \right) \end{cases} \rightarrow \delta = \frac{2}{3} \frac{k^2\Phi}{a^2 H^2} - \frac{3aHiv}{k}$$

$$\delta + ikv = -3\dot{\Phi} \rightarrow \frac{d}{d\eta} \left[\frac{2}{3} \frac{k^2\Phi}{a^2 H^2} - \frac{3aHiv}{k} \right] + ikv = -3\dot{\Phi}$$

Let's see if $\Phi = \text{const}$ is a solution! \rightarrow prove $\Phi = \text{const}$

$$\frac{d}{d\eta} \left[\frac{2}{3} \frac{k^2\Phi}{a^2 H^2} - \frac{3aHiv}{k} \right] + ikv = 0 \quad \frac{d[(aH)^{-2}]}{d\eta} = \frac{d}{d\eta} \left(\frac{\eta^2}{4} \right) = \frac{\eta}{2} = \frac{1}{aH}$$

Useful relations during MD

$$aH = \frac{2}{\eta} \rightarrow \frac{d(aH)}{d\eta} = -\frac{2}{\eta^2} = -\frac{a^2 H^2}{2} \quad \frac{d[(aH)^{-1}]}{d\eta} = \frac{1}{2} \quad \frac{d[(aH)^2]}{d\eta} = 2aH \frac{d(aH)}{d\eta} = -a^2 H^3$$

So we just need to expand

$$\frac{d}{d\eta} \left[\frac{2}{3} \frac{k^2 \Phi}{a^2 H^2} - \frac{3aH\dot{v}}{k} \right] + ikv = 0$$

Use the velocity equation to eliminate \dot{v} , and check if $\ddot{\Phi} = 0$ is a solution

$$\frac{2k^2}{3a^2 H^2} \dot{\Phi} + \frac{2}{3} k^2 \Phi \frac{d[(aH)^{-2}]}{d\eta} - \frac{3aH\dot{v}}{k} \Rightarrow \frac{3\dot{v}}{k} \frac{d(aH)}{d\eta} + ikv =$$

$\underbrace{\hspace{10em}}_{VaH}$
 $\underbrace{\hspace{10em}}_{-a^2 H^2}$

$$= \frac{2k^2 \dot{\Phi}}{3a^2 H^2} + \frac{2k^2 \Phi}{3aH} - \frac{3aH\dot{v}}{k} + \frac{3a^2 H^2 \dot{v}}{2k} + ikv =$$

$\dot{v} = ik\Phi - \frac{d}{d\eta}v = ik\Phi - aHv$

$$= \frac{2k^2 \dot{\Phi}}{3a^2 H^2} + \frac{2k^2 \Phi}{3aH} + 3aH\Phi + \frac{3a^2 H^2 \dot{v}}{k} + \frac{3a^2 H^2 \dot{v}}{2k} + ikv =$$

$$= \frac{2k^2 \dot{\Phi}}{3a^2 H^2} + \frac{2k^2 \Phi}{3aH} + 3aH\Phi + \frac{9}{2} \frac{a^2 H^2 \dot{v}}{k} + ikv =$$

$$= \left[\frac{2k^2 \dot{\Phi}}{3a^2 H^2} + \left(\frac{\dot{v}}{k} + \frac{2\Phi}{3aH} \right) \left(\frac{9}{2} a^2 H^2 + k^2 \right) = 0 \right]$$

differential equation for Φ

large-scale, sub-horizon modes

Take derivatives in η , use velocity equation to eliminate $\dot{v} \rightarrow$ ideally get equation for $\ddot{\Phi}$

Goal = show that equation is of the form

$$\alpha \ddot{\Phi} + \beta \dot{\Phi} = 0$$

Prove that terms other than $\ddot{\Phi}, \dot{\Phi}$ are 0!

$\hookrightarrow \Phi = \text{const}$ solution.

$$\frac{d}{d\eta} \left[\frac{2k^2 \dot{\Phi}}{3a^2 H^2} + \frac{2k^2 \dot{\Phi}}{3aH} + 3aH \dot{\Phi} + \frac{q}{2} \frac{a^2 H^2 i\dot{v}}{k} + ikv \right] = \text{look only at no-derivatives } \Phi \text{ terms}$$

$$= \frac{2k^2 \ddot{\Phi}}{3a^2 H^2} + \frac{2k^2 \dot{\Phi}}{3} \frac{d[(aH)^2]}{d\eta} + \frac{2}{3aH} k^2 \ddot{\Phi} + \frac{2k^2 \dot{\Phi}}{3} \frac{d[(aH)^2]}{d\eta} + 3aH \ddot{\Phi} + 3\dot{\Phi} \frac{d(aH)}{d\eta}$$

$$+ \frac{q}{2} \frac{a^2 H^2 i\ddot{v}}{k} + \frac{q}{2} \frac{i\dot{v}}{k} \frac{d[(aH)^2]}{d\eta} + ik\dot{v} =$$

$$\text{---} \frac{1}{2} \text{---} \text{---} -a^2 H^2 / 2$$

$$= \frac{k^2 \ddot{\Phi}}{3} - \frac{3a^2 H^2 \dot{\Phi}}{2} + \frac{q}{2} \frac{a^2 H^2 i\ddot{v}}{k} - \frac{q}{2} \frac{a^3 H^3 i\dot{v}}{k} + ik\dot{v} =$$

$$\text{---} -a^3 H^3 \text{---} \text{---} \dot{v} = ik\dot{\Phi} - aHv$$

$$= \frac{k^2 \ddot{\Phi}}{3} - \frac{3a^2 H^2 \dot{\Phi}}{2} - \frac{q}{2} \frac{a^3 H^3 \dot{\Phi}}{k} - \frac{q}{2} \frac{a^3 H^3 i\dot{v}}{k} - \frac{q}{2} \frac{a^3 H^3 i\dot{v}}{k} - k^2 \ddot{\Phi} - ik a H v =$$

$$= -\frac{2}{3} k^2 \ddot{\Phi} - ik a H v - \frac{q}{2} \frac{a^3 H^3 i\dot{v}}{k} - 6a^2 H^2 \dot{\Phi} =$$

$$= -\left(\frac{ik a H v}{k} + \frac{2\dot{\Phi}}{3} \right) (9a^2 H^2 + k^2) \propto \ddot{\Phi}$$

since $\frac{2k^2 \ddot{\Phi}}{3a^2 H^2} + \left(\frac{i\dot{v}}{k} + \frac{2\dot{\Phi}}{3aH} \right) \left(\frac{9a^2 H^2}{2} + k^2 \right) = 0$

$$\propto \ddot{\Phi}$$

So Φ equation has only $\ddot{\Phi}$ and $\dot{\Phi}$, no Φ term

$$\alpha \ddot{\Phi} + \beta \dot{\Phi} = 0 \longrightarrow \Phi = \text{const is a solution}$$

But initial conditions pick up $\Phi = \text{const} \longrightarrow \Phi = \text{const}$ is the solution
 $\Phi \sim \text{const}$ as long as MD holds, then decay due to DE, but this decay is described by $\Omega(a)$

Conclusion: for $k \ll a_{eq} H(a_{eq}) \simeq 0.073 \Omega_m h^2 \text{ Mpc}^{-1}$

• $T(k) \sim 1$

Transfer function is very close to 1!

$T(k) = T\left(\frac{k}{k_{eq}}\right)$

since $y = \frac{a}{a_{eq}}$

large-scale (M) probes initial conditions $\rightarrow \Phi_p(M)$

Note: $\Phi \sim \text{const} \rightarrow \delta \propto a$

Poisson equation

$k^2 \Phi \sim 4\pi G a^2 \rho_{dm} \delta \propto a^2 a^{-3} \delta = \frac{\delta}{a} = \text{const}$

\uparrow
due to a^{-3}

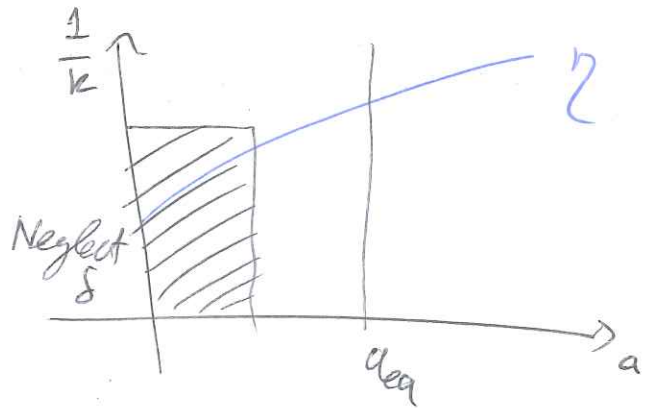
$\Rightarrow \boxed{\delta \propto a}$ during MD era

Small scales

Modes cross deep during RD, then to MD once sub-horizon

Horizon crossing

• Φ mostly determined by ρ which influences δ but is not influenced by it



$\Theta_0, \Theta_1 \rightleftharpoons \Phi \longrightarrow \delta, v$ " \longrightarrow " = influences

Strategy:

• solve for Θ_0, Θ_1, Φ

• solve for δ using Φ as external driving force