So ζ goes from $\simeq -\frac{aH\delta\phi}{\dot\phi^{(0)}}$ to $\simeq -\frac{3}{2}\Psi$

before and around horizon crossing

well after the end of inflation, on superhorizon scales

Why is ζ important? It is conserved for super-horizon perturbations (show later)

Since it is conserved

$$-\frac{aH\delta\phi}{\dot\phi^{(0)}}\Big|_{\text{horizon crossing}} = -\frac{3}{2}\Psi\Big|_{\text{post inflation}}$$

$$\Rightarrow \Psi\Big|_{\text{post inflation}} = \frac{2aH}{3}\frac{\delta\phi}{\dot\phi^{(0)}}\Big|_{\text{horizon crossing}}$$

But we care about the variance of Ψ (power spectrum)

$$\Rightarrow P_\Psi = \frac{4}{9}\frac{a^2H^2}{\dot\phi^{(0)2}} P_{\delta\phi}\Big|_{\text{horizon crossing}}$$

where ~~has~~ a mode k crosses the horizon when $k = aH$

$$P_\Psi = \frac{4}{9}\left(\frac{aH}{\dot\phi^{(0)}}\right)^2 P_{\delta\phi}\Big|_{aH=k} \underset{\uparrow}{=} \frac{4}{9}\left(\frac{aH}{\dot\phi^{(0)}}\right)^2 \frac{H^2}{2k^3}\Big|_{aH=k} = \frac{2}{9}\left(\frac{aH^2}{\dot\phi^{(0)}}\right)^2\frac{1}{k^3}$$

$P_{\delta\phi} = \frac{H^2}{2k^3}$

Exercise: $\left(\frac{aH}{\dot\phi^{(0)}}\right)^2 \simeq \frac{4\pi G}{\varepsilon}$

$$\Rightarrow P_\Psi(k) \underset{\uparrow}{\simeq} P_\Phi(k) = \frac{2}{9}\frac{4\pi G}{\varepsilon}\frac{H^2}{k^3} = \frac{8\pi G}{9k^3}\frac{H^2}{\varepsilon}\Big|_{aH=k}$$

$\Psi \simeq -\Phi$

Recall $P_h \sim \frac{8\pi G H^2}{k^3} \longrightarrow \frac{P_{\Psi/\Phi}}{P_h} \sim \frac{1}{\varepsilon} \gg 1$

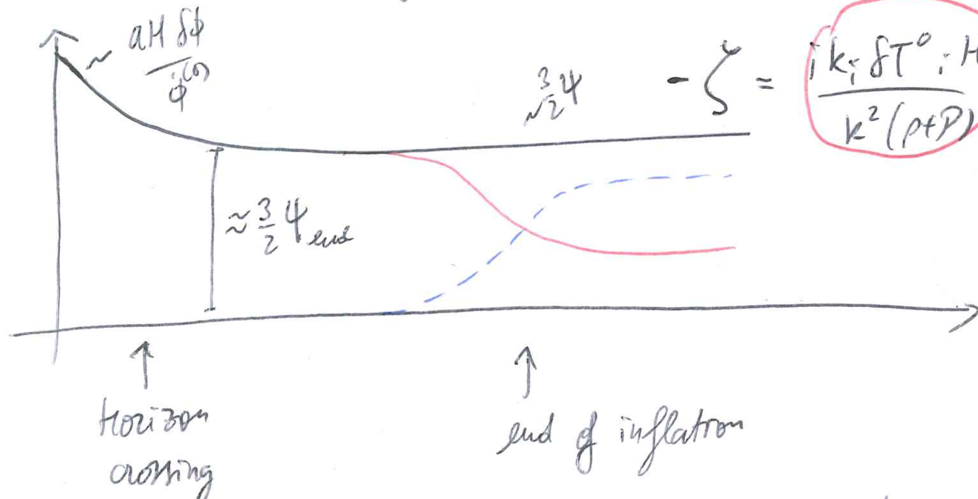
So we expect scalar modes to dominate over tensor modes in slow-roll inflation

Exercise: $\varepsilon \simeq \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2$

$$\Rightarrow P_\Psi(k) = P_\zeta(k) = \frac{8\pi G}{9k^3}\frac{H^2}{\varepsilon}\Big|_{aH=k} \simeq \frac{128\pi^2 G^2}{9k^3}\left(\frac{H^2 V^2}{V'^2}\right)\Big|_{aH=k}$$

Conservation of ζ for super-horizon modes, schematically

$$-\zeta = \frac{ik_i\,\delta T^0{}_i\,H}{k^2(\rho+P)} + \Psi$$



Sub-horizon: QM fluctuations set up in $\phi^{(0)} \Rightarrow \delta\phi \Rightarrow \frac{ik_i\,\delta T^0{}_i\,H}{k^2(\rho+P)} \simeq \frac{aH\,\delta\phi}{\dot\phi^{(0)}}$

Negligible perturbations to the metric

Past horizon crossing: ζ conserved, but relative contribution of Ψ to $\delta T^0{}_i$ increases, finally $\zeta \approx -\frac{3}{2}\Psi$

Now we just need to show that ζ is indeed conserved on super-horizon scales. Go back to conservation equation (optional)

$\delta T^i{}_0 = ik_i\,\dot\phi^{(0)}\delta\phi/a^3 \sim k_i \Rightarrow ik_i\,\delta T^i{}_0 \sim k^2$ negligible on large scales

$$\frac{\partial\,\delta T^0{}_0}{\partial t} + \cancel{ik_i\,\delta T^i{}_0} + 3H\,\delta T^0{}_0 - H\,\delta T^i{}_i = -3(P+\rho)\frac{\partial\Psi}{\partial t}$$

On large scales (exercise) $\frac{ik_i\,\delta T^0{}_i\,H}{k^2} \simeq \frac{\delta T^0{}_0}{3}$

$$\Rightarrow \zeta = \frac{ik_i\,\delta T^0{}_i\,H}{k^2(\rho+P)} - \Psi \simeq -\Psi - \frac{1}{3}\frac{\delta T^0{}_0}{\rho+P}$$

large scales

Combine the two

$$\begin{cases} \dfrac{\partial \delta T^0{}_0}{\partial t} + 3H\delta T^0{}_0 - H\delta T^i{}_i = -3(P+\rho)\dfrac{\partial \psi}{\partial t} \\[2ex] \zeta \simeq -\psi - \dfrac{1}{3}\dfrac{\delta T^0{}_0}{\rho+P} \quad \rightarrow \psi \simeq -\zeta - \dfrac{1}{3}\dfrac{\delta T^0{}_0}{\rho+P} \end{cases}$$

$$\hookrightarrow \cancel{\frac{\partial \delta T^0{}_0}{\partial t}} + 3H\delta T^0{}_0 - H\delta T^i{}_i = -3(P+\rho)\frac{\partial}{\partial t}\left[-\zeta - \frac{1}{3}\frac{\delta T^0{}_0}{\rho+P}\right] =$$

$$= 3(P+\rho)\frac{\partial}{\partial t}\left[\zeta + \frac{1}{3}\frac{\delta T^0{}_0}{\rho+P}\right] = 3(P+\rho)\frac{\partial \zeta}{\partial t} + (P+\rho)\frac{\partial}{\partial t}\left[\frac{\delta T^0{}_0}{P+\rho}\right] =$$

$$= 3(P+\rho)\frac{\partial \zeta}{\partial t} + \cancel{\frac{\partial \delta T^0{}_0}{\partial t}} - \delta T^0{}_0 \frac{\cancel{(P+\rho)}}{(P+\rho)^2}\left(\frac{d\rho}{dt} + \frac{dP}{dt}\right)$$

$$\Rightarrow \delta T^0{}_0\left[3H + \frac{1}{\rho+P}\left(\frac{d\rho}{dt} + \frac{dP}{dt}\right)\right] - H\delta T^i{}_i = 3(P+\rho)\frac{\partial \zeta}{\partial t}$$

Continuity equation

$$\dot\rho + 3H\rho(1+w) = 0 \Rightarrow \frac{d\rho}{dt} + 3H\rho\left(1 + \frac{P}{\rho}\right) = 0$$

$$\Rightarrow \frac{d\rho}{dt} = -3H(\rho+P)$$

so $\cancel{3H} + 3H + \dfrac{1}{\rho+P}\left(\dfrac{d\rho}{dt} + \dfrac{dP}{dt}\right) = \cancel{3H} - \cancel{3H} + \dfrac{1}{\rho+P}\dfrac{dP}{dt}$

$$\Rightarrow \frac{\delta T^0{}_0}{\rho+P}\frac{dP}{dt} - H\delta T^i{}_i = 3(P+\rho)\frac{\partial \zeta}{\partial t}$$

$$\Rightarrow \frac{\partial \zeta}{\partial t} = -\frac{1}{3(\rho+P)^2}\left[H(\rho+P)\delta T^i{}_i - \delta T^0{}_0\frac{dP}{dt}\right]$$

$H(\rho+P) = -\frac{1}{3}\frac{d\rho}{dt}$ from continuity equation

$$\Rightarrow [\ldots] \propto -\frac{1}{3}\frac{d\rho}{dt}\delta T^i{}_i - \delta T^0{}_0\frac{dP}{dt} \propto \frac{\delta T^i{}_i}{3} + \delta T^0{}_0\frac{dP}{d\rho} \propto \delta P - \frac{dP}{d\rho}\delta\rho$$

$\delta T^i{}_i = 3\delta P$ $\delta T^0{}_0 = -\delta\rho$

so $\frac{\partial \zeta}{\partial t} \propto \delta P - \frac{dP}{d\rho}\delta\rho = 0$ for adiabatic perturbations!

~~Example~~: Recall for adiabatic perturbations the number density ratio between species is identical everywhere

so $\frac{\partial \zeta}{\partial t} = 0$ on super-horizon scales!

Spatially flat slicing

A cleaner way to obtain all the previous results is to use gauge transformations and gauge-invariant variables

In Newtonian gauge $\delta\phi$ and Ψ are coupled

Spatially flat gauge

$$ds^2 = -(1+2A)\,dt^2 + 2a\frac{\partial B}{\partial x^i}dx^i dt + \delta_{ij}a^2 dx^i dx^j$$

In this gauge the equation we found previously are exact with no approximations required

$$\ddot{\delta\phi} + 2aH\,\dot{\delta\phi} + k^2\delta\phi = 0 \implies P_{\delta\phi} = \frac{H^2}{2k^3}$$

Bardeen identified two gauge-invariant variables

Bardeen's velocity (gauge-invariant!)

~~$v \equiv ikB - ik\frac{\dot\phi^{(0)}\delta\phi}{(\rho+P)a^2}$ in spatially flat slicing~~

$$v \equiv ikB + \frac{\hat{k}^i T^0{}_i}{(\rho+P)a} = ikB - ik\frac{\dot\phi^{(0)}\delta\phi}{(\rho+P)a^2} \quad \text{in spatially flat slicing}$$

$\uparrow \delta T^0{}_i = \frac{ik_i}{a}\dot\phi\,\delta\phi$

Bardeen's potential

$$\Phi_H = -\Psi + aH(B - \dot{E}) = aHB \quad (\Psi = 0,\ E = 0)$$

Any linear combination of Φ_H and v is gauge-invariant

Take: $\zeta \equiv -\Phi_H - \frac{iaH}{k} v = -aHB - \frac{iaH}{k} ikB - \frac{aH\dot{\phi}^{(0)}\delta\phi}{(\rho+P)a^2} =$

$$= \frac{-aH\dot{\phi}^{(0)}\delta\phi}{(\rho+P)a^2} \simeq \frac{-aH\,\delta\phi}{\dot{\phi}^{(0)}}$$

$\rho+P = \left(\frac{\dot{\phi}^{(0)}}{a}\right)^2$

$$\Rightarrow P_\zeta = \left(\frac{aH}{\dot{\phi}^{(0)}}\right)^2 P_{\delta\phi} = \frac{4\pi G}{\varepsilon}\frac{H^2}{2k^3} = \left.\frac{2\pi G H^2}{\varepsilon k^3}\right|_{aH=k}$$

$P_{\delta\phi} = \frac{H^2}{2k^3}$, $\left(\frac{aH}{\dot{\phi}^{(0)}}\right)^2 = \frac{4\pi G}{\varepsilon}$

power spectrum of a gauge-invariant quantity!!!

In conformal Newtonian gauge

$\Phi_H = -\Phi$ so $\zeta = -\Phi_H - \frac{iaH}{k} v = \frac{-ik_i\delta T^0{}_i H}{k^2(\rho+P)} - \Psi$ as we saw earlier

We also saw that after inflation $\zeta \simeq -\frac{3}{2}\Psi \simeq \frac{3}{2}\Phi$

$$\Rightarrow P_\Phi = \frac{4}{9}P_\zeta = \left.\frac{8\pi G H^2}{9\varepsilon k^3}\right|_{aH=k}$$ ✓ Matches previous result!

Physical interpretation of Φ_H: $\frac{4k^2\Phi_H}{a^2}$ is curvature of 3D space at fixed time $\rightarrow$ Perturbations to Φ_H are curvature perturbations (even though space-time flat to 0th order)

In a comoving gauge $v=0 \rightarrow \zeta = \Phi_H$ so ζ shares an interpretation as curvature: scalar perturbations from inflation often called curvature perturbations!

Spectral indices

So far

$\langle \Phi(\bar{k}) \rangle = 0$ $\quad \langle \Phi(\bar{k})\, \Phi^*(\bar{k}') \rangle = (2\pi)^3 P_\Phi(k)\, \delta^3(\bar{k}-\bar{k}')$

$$P_\Phi(k) = \left.\frac{8\pi G H^2}{9\varepsilon k^3}\right|_{aH=k} \qquad P_h(k) = \frac{8\pi G H^2}{k^3}$$

$P_\Phi \propto \frac{1}{\varepsilon}$ where $\varepsilon = -\frac{\dot{H}}{aH^2} \ll 1$ during inflation

quasi de-Sitter $\Downarrow$ quasi scale-invariant

- $k^3 P_\Phi(k) \approx \text{const} \rightarrow$ scale invariant

However in reality ~~there~~ there are small deviations from scale-invariance:
(because inflation is not perfectly de Sitter $\Rightarrow \eta = -\frac{1}{aH}$: inflation has to end!)

$$P_\Phi(k) = \frac{8\pi}{9k^3}\frac{H^2}{\varepsilon M_{Pl}^2}\Big|_{aH=k} \equiv \frac{50\pi^2}{9k^3}\left(\frac{k}{H_0}\right)^{n_s-1}\delta_H^{\;2}\underbrace{\left(\frac{\Omega_m}{D_1(a=1)}\right)^2}_{\text{later, growth function}}$$

$$P_h(k) = \frac{8\pi H^2}{k^3 M_{Pl}^2}\Big|_{aH=k} \equiv A_T\, k^{n_T-3}$$

- δ_H , A_T scalar and tensor amplitudes
  n_s , n_T scalar and tensor tilts $\rightarrow$ scale-invariance $n_s=1$, $n_T=0$

n_s and n_T are related to the slow-roll parameters

Since $P_h(k) = A_T k^{n_T-3} \Rightarrow \frac{d\ln P_h}{d\ln k} = n_T - 3$ $\left[\frac{k}{P_h}\frac{dP_h}{dk} = \frac{\cancel{k}}{\cancel{A_T k^{n_T-3}}}(n_T-3)\cancel{A_T} k^{n_T-2}\right]$

$\ln P_h(k) = \ln(A_T) + (n_T-3)\ln k$

- $n_T - \cancel{3} = \frac{d\ln P_h}{d\ln k} = \frac{d}{d\ln k}\left[\ln\left(\frac{H^2}{2k^3}\right)\right] = \frac{d}{d\ln k}\left[2\ln H - 3\ln k\right] = -\cancel{3} + 2\frac{d\ln(H)}{d\ln(k)}$

$\Rightarrow n_T = 2\frac{d\ln(H)}{d\ln(k)}$

$\frac{d\ln(H)}{d\ln(k)}\Big|_{aH=k} = \frac{k}{H}\frac{dH}{dk} = \frac{k}{H}\frac{dH}{d\eta}\frac{d\eta}{dk}\Big|_{aH=k} =$

$\dot{H} \equiv -aH^2\varepsilon \longrightarrow \frac{dH}{d\eta} = -aH^2\varepsilon$

$= -\frac{k}{H}\frac{aH^2\varepsilon}{k^2}\Big|_{aH=k} =$

$\frac{d\eta}{dk}\Big|_{aH=k} = -\frac{d(aH)^{-1}}{dk}\Big|_{aH=k} = -\frac{d}{dk}\left(\frac{1}{k}\right) = \frac{1}{k^2}$

$\uparrow \eta \simeq -\frac{1}{aH}$

$= -\frac{aH}{k}\varepsilon\Big|_{aH=k} = -\varepsilon$

-

$\Rightarrow \boxed{n_T = 2\frac{d\ln(H)}{d\ln(k)} = -2\varepsilon}$ So expect very small tensor tilt $n_T \ll 1$

Similarly for n_s

$$n_s - 1 = \frac{d \ln P_{\Phi}}{d \ln k} = \frac{d}{d\ln k}\left[\ln(H^2) - \ln(\varepsilon)\right]$$

$$\frac{d\ln(H^2)}{d\ln k} = 2\,\frac{d\ln H}{d\ln k} = -2\varepsilon \qquad \frac{d\ln(\varepsilon)}{d\ln k} = -2(\varepsilon + \delta) \quad \text{exercise!}$$

$$\Rightarrow n_s - 1 \simeq -2\varepsilon - 2\varepsilon + 2\delta \Rightarrow \boxed{n_s \simeq 1 - 4\varepsilon + 2\delta}$$

Note $\frac{P_{\Phi}}{P_h} \sim \frac{1}{\varepsilon}$ and $n_T \sim \varepsilon$: this is a robust prediction of single-field slow-roll inflation!

~~Worked~~ Tensor-to-scalar ratio : $r \equiv \frac{\Delta_h^2}{\Delta_{\Phi}^2} \simeq 16\varepsilon$

Summary : inflationary observables (see 0907.5424 arXiv)

$$\varepsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{\dot{H}}{aH^2} = -\frac{d\ln H}{dN} < 1 \quad \text{where } dN = d\ln a = H dt$$

↳ number of e-folds

$$\delta \equiv \frac{1}{H}\,\frac{d^2\phi^{(0)}/dt^2}{d\phi^{(0)}/dt} = \varepsilon - \frac{1}{2\varepsilon}\frac{d\varepsilon}{d\eta} \qquad |\eta| < 1$$

de Sitter limit : $\varepsilon \to 0$ $\left[\frac{\ddot{a}}{a} = H^2(1-\varepsilon)\right]$

$\varepsilon \ll 1$: potential very flat $|\eta| \ll 1$: ~~change~~ fractional change of ε per e-fold small

Potential slow-roll parameters

$$\varepsilon_V(\phi) \equiv \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2$$

$$\eta_V(\phi) \equiv M_{pl}^2\,\frac{V''}{V}$$

slow-roll: $\varepsilon_V, |\eta_V| \ll 1$

In the slow-roll regime

$\varepsilon \approx \varepsilon_V \qquad \eta \approx \eta_V - \varepsilon_V \qquad$ (Exercise) $\qquad H^2 \approx \frac{V(\phi)}{3} \simeq \text{const}$

$\dot\phi \approx -\frac{V'}{3H}$

$\hookrightarrow (|\ddot\phi| \ll |3H\dot\phi|, |V'|)$

Inflation ends when

$\varepsilon(\phi_{end}) \equiv 1 \qquad \varepsilon_V(\phi_{end}) \approx 1$

$$N(\phi) \equiv \ln \frac{a_{end}}{a} = \int_t^{t_{end}} dt\, H = \int_\phi^{\phi_{end}} \frac{dt}{d\phi} d\phi\, H = \int_\phi^{\phi_{end}} d\phi \frac{H}{\dot\phi} \approx \int_{\phi_{end}}^{\phi} d\phi \frac{V}{V'} =$$

$\uparrow$ number of e-folds before inflation ends

$$= \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\varepsilon}} \approx \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\varepsilon_V}} \gtrsim 60$$

Fluctuations observed in the CMB exited horizon $\approx 40-60$ e-folds before the end of inflation

$$\int_{\phi_{end}}^{\phi_{cmb}} \frac{d\phi}{\sqrt{2\varepsilon_V}} = N_{cmb} \approx 40-60$$

$n_s - 1 \simeq 2\eta_V^* - 6\varepsilon_V^*$

$n_T \simeq -2\varepsilon_V^*$

$r \simeq 16\varepsilon_V^*$

$\hookrightarrow r = -8 n_T$ consistency condition

$*$: when CMB modes exited horizon

<u>Worked example</u>

$m^2\phi^2$ inflation driven by standard mass term of scalar field

$V(\phi) = \frac{1}{2} m^2 \phi^2 \qquad V'(\phi) = m^2\phi \qquad V''(\phi) = m^2$

$\varepsilon_V(\phi) = \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right) = 2\left(\frac{M_{Pl}}{\phi}\right)^2 \qquad \eta_V(\phi) = M_{Pl}^2 \frac{V''}{V} = 2\left(\frac{M_{Pl}}{\phi}\right)^2 = \varepsilon_V(\phi)!$

$\varepsilon_V, |\eta_V| < 1 \Longrightarrow \phi \gtrsim \sqrt{2} M_{Pl}$ inflation ends when $\phi \approx \sqrt{2} M_{Pl}$ $(\varepsilon_V \approx 1, |\eta_V| \approx 1)$

$$N(\phi) = \int_{\phi_{end}}^{\phi} \frac{d\phi'}{\sqrt{2\varepsilon_V(\phi)}} = \int_{\phi_{end}}^{\phi} \frac{d\phi'}{2M_{Pl}} \phi' = \int_{\sqrt{2}M_{Pl}}^{\phi} \frac{d\phi'\phi'}{2M_{Pl}} = \frac{\phi'^2}{4M_{Pl}}\Bigg|_{\sqrt{2}M_{Pl}}^{\phi} = \frac{\phi^2}{4M_{Pl}^2} - \frac{1}{2}$$

$$N(\phi_{CMB}) \sim 60 \Rightarrow \frac{\phi^2}{4M_{Pl}^2} - \frac{1}{2} \simeq 60 \rightarrow \phi_{CMB} \sim 2\sqrt{N_{CMB}}\, M_{Pl} \sim 15 M_{Pl}$$

so fluctuations observed in the CMB are created when ϕ is well super-Planckian

$$\phi_{CMB} = 2\sqrt{N_{CMB}}\, M_{Pl} \Rightarrow \varepsilon_V^* = \eta_V^* = 2\left(\frac{M_{Pl}}{\phi_{CMB}}\right)^2 = \frac{1}{2N_{CMB}}$$

$$n_s = 1 + 2\eta_V^* - 6\varepsilon_V^* \simeq 1 - \frac{2}{N_{CMB}} \simeq 0.96$$

in excellent agreement with observations

$n_s \sim 0.965 \pm 0.004$ (Planck 2018)

$$r = 16\,\varepsilon_V = \frac{8}{N_{CMB}} \approx 0.1$$

excluded by observations

$r \lesssim 0.07$ (Planck + BICEP)

(Alternative clearer presentation of horizon and flatness problem)

- $$\eta = \int_0^a \frac{da'}{Ha'^2} \propto \begin{cases} a & RD \longrightarrow a(t) \propto t^{1/2}, \ (aH)^{-1} \propto t^{1/2} \\ a^{1/2} & MD \longrightarrow a(t) \propto t^{2/3}, \ (aH)^{-1} \propto t^{1/3} \end{cases}$$

η grows monotonically with time so at any given time a scale entering the horizon was super-horizon when the CMB formed

- $$H^2 = \frac{\rho(a)}{3} - \frac{k}{a^2} \qquad (8\pi G = 1)$$

$$\Rightarrow 1 - \Omega(a) = -\frac{k}{(aH)^2} \qquad \Omega - 1 = \frac{\rho - \rho_{crit}}{\rho_{crit}}$$

~~In MD, RD~~ But $\frac{1}{(aH)^2}$ always grows in MD, RD

So $\Omega \sim 1$ today requires extreme fine-tuning in the past!

INHOMOGENEITIES

- Now we want to explicitly solve the Einstein-Boltzmann system with initial conditions provided by inflation

Recall what is the system and initial conditions, neglecting polarization and assuming massless neutrinos

(recall also $\delta_\gamma \sim 4\Theta_0$, $\delta_\nu \sim 4\mathcal{N}_0$, $v_\gamma \sim -3i\Theta_1$, $v_\nu \sim -3i\mathcal{N}_1$)

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_b\right] \quad \} \gamma$$

$\dot{\tau} = -n_e \sigma_T a$

$$\dot{\delta}_{dm} + ikv_{dm} = -3\dot{\Phi}$$

$$\dot{v}_{dm} + \frac{\dot{a}}{a} v_{dm} = -ik\Psi \quad \} \text{DM}$$

$\dot{} \equiv \frac{d}{d\eta} = a\frac{d}{dt}$

$\Theta_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1} \frac{d\mu}{2} \mathcal{P}_\ell(\mu)\Theta(\mu)$

$$\dot{\delta}_b + ikv_b = -3\dot{\Phi}$$

$$\dot{v}_b + \frac{\dot{a}}{a} v_b = -ik\Psi + \frac{\dot{\tau}}{R}\left[v_b + 3i\Theta_1\right] \quad \} b$$

$R = \frac{3\rho_b^{(0)}}{4\rho_\gamma^{(0)}}$

$\mathcal{N}_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1} \frac{d\mu}{2} \mathcal{P}_\ell(\mu)\mathcal{N}(\mu)$

$$\dot{\mathcal{N}} + ik\mu\mathcal{N} = -\dot{\Phi} - ik\mu\Psi \quad \} \nu$$

- $$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) = 4\pi G a^2\left(\rho_{dm}\delta_{dm} + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right)$$

$$k^2(\Phi + \Psi) = -32\pi G a^2\left(\rho_\gamma\Theta_2 + \rho_\nu\mathcal{N}_2\right)$$

$\} \Phi, \Psi$ scalar/curvature perturbations to the metric

$$\ddot{h}_+ + 2\frac{\dot{a}}{a}\dot{h}_+ + k^2 h_+ = 0$$

$$\ddot{h}_\times + 2\frac{\dot{a}}{a}\dot{h}_\times + k^2 h_\times = 0$$

$\} h_+, h_\times$ tensor perturbation to the metric/GWs

$$\Phi(k,\eta_i) = -\Psi(k,\eta_i) = 2\Theta_0(k,\eta_i) = 2\mathcal{N}_0(k,\eta_i)$$

$$\delta_{dm}(k,\eta_i) = \delta_b(k,\eta_i) = 3\Theta_0(k,\eta_i) = \frac{3}{2}\Phi(k,\eta_i) = -\frac{3}{2}\Psi(k,\eta_i)$$

- $$\Theta_1(k,\eta_i) = \mathcal{N}_1(k,\eta_i) = \frac{iv_{dm}(k,\eta_i)}{3} = \frac{iv_b(k,\eta_i)}{3} = \frac{-k\Phi(k,\eta_i)}{6aH}$$

η_i such that $k\eta_i \ll 1$ for all modes of interest, but well after inflation ended

Our goal: figure out $\underbrace{\text{inhomogeneities}}_{\text{late-Universe matter}}$ and $\underbrace{\text{anisotropies}}_{\text{CMB}}$

We start from inhomogeneities, looking at evolution of perturbations to the DM. Simple picture:

Early times	Late times
Radiation $\sim \Theta_0, \Theta_1, \mathcal{N}_0, \mathcal{N}_1$	Radiation negligible
$\downarrow$	
affects Φ, Ψ	Φ, Ψ
$\downarrow$	$\downarrow$
affects δ, v (indirectly)	affect δ, v

We want to solve for the evolution of each Fourier mode $\delta_{dm}(k, \eta)$

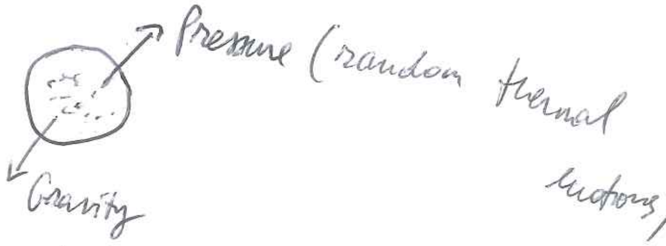
[from now on $\underline{\delta_{dm} \longrightarrow \delta}$]

Knowing $\delta(k,\eta)$, with initial conditions generated by inflation, we can compute (dark) matter power spectrum today, compare to observations (at least on large scales)

<u>Solution strategy</u>

Basic idea of gravitational instability [sub-horizon]

$$\ddot{\delta} + [\text{Pressure} - \text{Gravity}]\, \delta = 0$$



<u>Any</u> initial overdensity will eventually grow

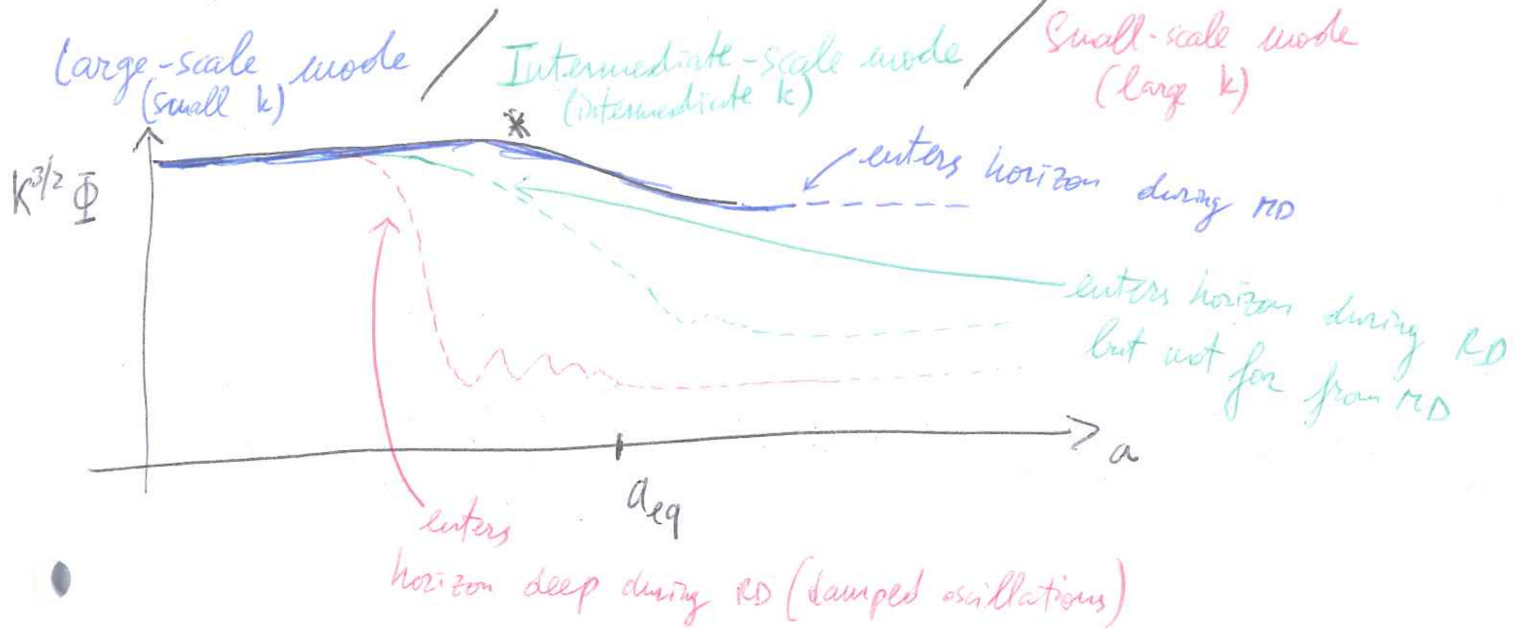
Schematically:

- large P, δ oscillates
- low P, δ grows (exponentially) $\longrightarrow$ actually $\begin{cases} \delta \propto t & \text{MD} \\ \delta \propto \ln(t) & \text{RD} \end{cases}$

Three natural stages of evolution (a posteriori)

- Early times: mode is super-horizon, $\Phi \sim$ const ($k\eta \ll 1$)
- Intermediate times: mode enters horizon <u>and</u> radiation-to-matter domination → what happens to the mode depends on whether it enters during RD (damped oscillations) or MD ($\sim$ const). Larger scale modes enter the horizon later (during MD rather than RD) so we expect suppression due to radiation pressure on small scales
- Late times: matter domination, $\Phi \sim$ const

Example, schematic plot



Schematically

$$\Phi(\bar{k},a) = \underbrace{\Phi_p(\bar{k})}_{\text{inflation}} \times \underbrace{\{\text{Transfer function}(k)\}}_{\substack{\text{evolution through horizon crossing}\\ \text{and matter domination}\\ \sim 1 \text{ on large scales}}} \times \underbrace{\{\text{Growth function}(a)\}}_{\substack{k\text{-independent}\\ \text{late-time growth}}}$$

$$T(k) \equiv \frac{\Phi(k, a_{late})}{\Phi_{large\,scale}(k, a_{late})}$$

where a_{late} well in matter domination

$\Phi_{large\,scale}$ decreased by $\sim \frac{9}{10}$ relative to Φ_p *

Growth function D_1

$$\frac{D_1(a)}{a} \equiv \frac{\Phi(a)}{\Phi(a_{late})} \qquad (a > a_{late})$$

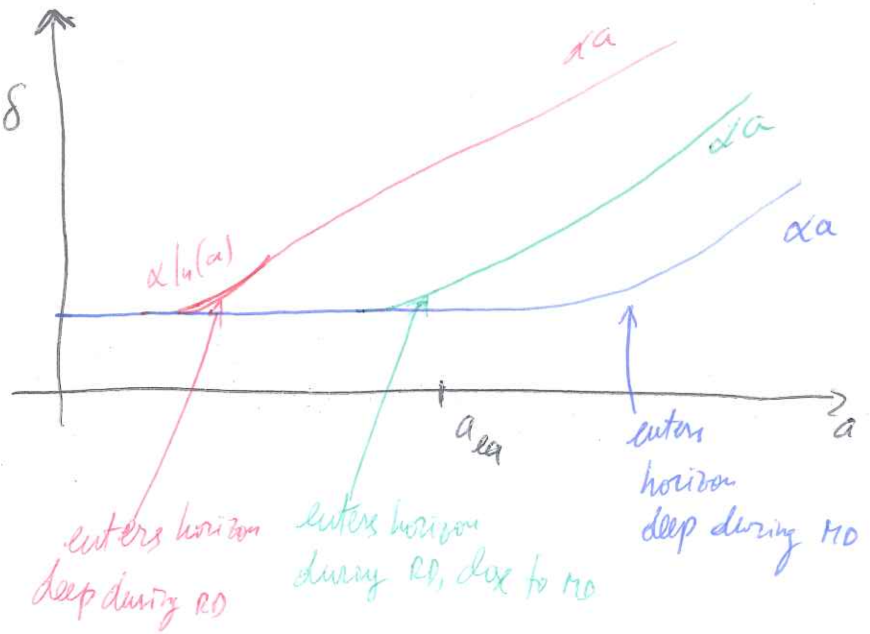
Because for $\Omega_\Lambda = 0, \Omega_m = 1$, $\Phi \sim const \Rightarrow D_1 \propto a$

$$\Rightarrow \boxed{\Phi(\bar{k}, a) = \frac{9}{10}\Phi_p(\bar{k})\, T(k)\, \frac{D_1(a)}{a}} \qquad (a > a_{late})$$

How to probe Φ? Measuring matter distribution!

Late times: $\delta \propto D_1(a) \propto a$

Same modes as before



We want matter power spectrum $P = P_\delta$ in terms of Φ_p

Use Poisson equation

$$k^2\Phi = 4\pi G a^2 \left[\rho_m \delta_m + 4\rho_r \Theta_{r,0} + \frac{3aH}{k}\left(i\rho_m v_m + 4\rho_r \Theta_{r,1}\right)\right] \xrightarrow[a > a_{late}]{\text{large-}k \text{ no radiation}} \Phi \approx \frac{4\pi G \rho_m a^2 \delta}{k^2}$$

$$\rho_m = \frac{\Omega_m \rho_{cr}}{a^3} \quad \text{and} \quad H_0^2 = \frac{8\pi G}{3}\rho_{cr} \rightarrow 4\pi G \rho_{cr} = \frac{3}{2}H_0^2$$

$$\Rightarrow \delta(\bar{k}, a) = \frac{k^2 \Phi(\bar{k}, a)}{4\pi G \rho_m a^2} = \frac{k^2\Phi(\bar{k}, a)\, a^{\cancel{3}}}{\Omega_m \rho_{cr} \cancel{a^2}\, 4\pi G} = \frac{k^2 \Phi(\bar{k}, a)\, a}{4\pi G \rho_{cr} \Omega_m} =$$

$$= \frac{k^2 \Phi(\bar{k},a) a}{3/2\, \Omega_m H_0^2} \implies \delta(\bar{k},a) \approx \frac{2}{3} \frac{k^2 \Phi(\bar{k},a)}{\Omega_m H_0^2} a \quad (a > a_{late})$$

But we know that $\Phi(\bar{k},a) = \frac{9}{10} \Phi_p(\bar{k}) T(k) \frac{D_1(a)}{a}$

$$\implies \delta(\bar{k},a) \approx \frac{2}{3} \frac{k^2 \Phi(\bar{k},a)}{\Omega_m H_0^2} a = \frac{\cancel{2}}{\cancel{3}} \frac{k^2 \cancel{a}}{\Omega_m H_0^2} \frac{\cancel{9}^{3}}{\cancel{10}_{5}} \Phi_p(\bar{k}) T(k) \frac{D_1(a)}{\cancel{a}} \approx$$

$$\implies \boxed{\delta(\bar{k},a) \approx \frac{3}{5} \frac{k^2}{\Omega_m H_0^2} \Phi_p(\bar{k}) T(k) D_1(a) \quad (a > a_{late})}$$

Regardless of how Φ_p was generated

Recall from inflation

$$P_\Phi(k) = \frac{50\pi^2}{9k^3} \left(\frac{k}{H_0}\right)^{n_s - 1} \delta_H^2 \left(\frac{\Omega_m}{D_1(a=1)}\right)^2$$

$\langle \Phi \rangle = 0 \qquad \langle \Phi^2 \rangle \sim P_\Phi(k)$

$$\implies P_\delta(k,a) = P_\delta = \frac{9}{25} \frac{k^4}{\Omega_m^2 H_0^4} T^2(k) D_1^2(a) P_\Phi(k) =$$

$$= \frac{\cancel{9}}{\cancel{25}} \frac{\cancel{k^4}}{\cancel{\Omega_m^2} H_0^4} T^2(k) D_1^2(a) \frac{\cancel{50}^{2}\pi^2}{\cancel{9k^3}} \frac{k^{n_s - \cancel{1}}}{H_0^{n_s - 1}} \delta_H^2 \frac{\cancel{\Omega_m^2}}{D_1^2(a=1)} = 2\pi^2 \delta_H^2 \frac{k^{n_s}}{H_0^{n_s+3}} T^2(k) \left(\frac{D_1(a)}{D_1(a=1)}\right)^2$$

$$\implies \boxed{P_\delta \approx 2\pi^2 \delta_H^2 \frac{k^{n_s}}{H_0^{n_s+3}} T^2(k) \left(\frac{D_1(a)}{D_1(a=1)}\right)^2 \quad (a > a_{late})}$$

$[P] = \text{length}^3 \quad [\text{Mpc}^3, \text{ or typically}, h^{-3}\text{Mpc}^3]$

Multiply by k^3 to get dimensionless quantity

$$\frac{d^3k\, P(k)}{(2\pi)^3}$$: excess power in bin of width dk centered at k

$$\frac{d^3k\, P(k)}{(2\pi)^3} = \frac{dk\, k^2\, 4\pi}{8\pi^3} P(k) = dk \frac{k^2}{2\pi^2} P(k) = \frac{dk}{k} \frac{k^3 P(k)}{2\pi^2} =$$

$$= d(\ln k) \frac{k^3 P(k)}{2\pi^2} \equiv d(\ln k)\, \Delta^2(k)$$

$$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}$$ dimensionless power spectrum

What does Δ^2 measure?

$\Delta \ll 1$: small inhomogeneities

$\Delta \sim 1$: inhomogeneities becoming non-linear

$\Delta \gg 1$: large, non-linear inhomogeneities

On horizon-sized scale today $(k \sim H_0)$ Harrison-Zeldovich-Peebles $(n_s = 1)$ spectrum has

$$P(k,1) = 2\pi^2 \delta_H^2 \frac{H_0^{1}}{H_0^{1+3}} \underbrace{T^2(k)}_{=1} \left(\cancel{\frac{D_1(1)}{D_1(1)}}\right)^2 = 2\pi^2 \frac{\delta_H^2}{H_0^3}$$

$$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2} = \frac{k^3}{2\pi^2} 2\pi^2 \frac{\delta_H^2}{H_0^3} = \delta_H^2$$ $k = H_0$

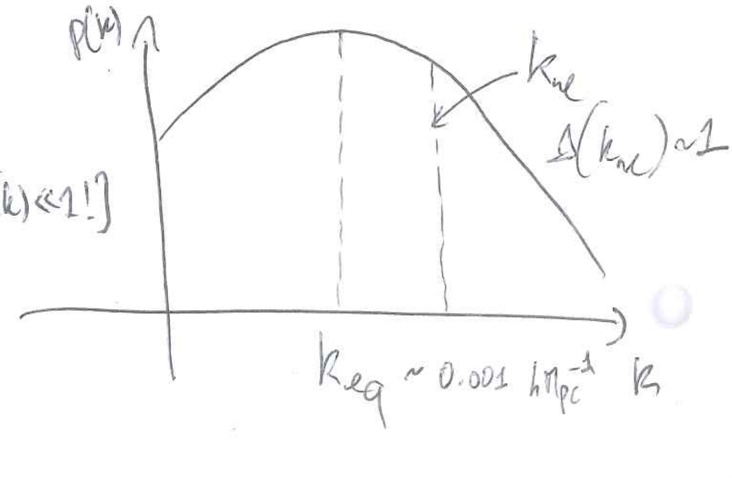
How do we expect $P(k)$ to be)

Large scales (small k) $T(k) \approx 1 \rightarrow P(k) \sim k^{n_s} \sim k$

Small scales (large k) $\rightarrow P(k)$ suppressed $[T(k) \ll 1!]$

modes enter horizon during RD

Suppressed by the radiation pressure



Strategy

We can significantly reduce the full set of Einstein-Boltzmann equations

- Simplifications:

  - before recombination photons characterized by Θ_0, Θ_1 only!
  - after recombination photons negligible (matter domination)

↳ we can neglect all radiation moments except monopole, dipole

Relevant equations

"r" = radiation γ and ν

$$\boxed{\dot{\Theta}_r + ik\mu\Theta_r = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_{0,r} - \Theta_r + \mu v_b\right]}$$

$$\dot{\delta} + ikv = -3\dot{\Phi}$$

$$\dot{v} + \frac{\dot{a}}{a}v = \cancel{ik\Phi} - ik\Psi$$

We can simplify these further, get equations for Θ_0 and Θ_1

- neglect baryons for the moment ~~...~~ $\longrightarrow \dot{\tau}[\ldots] = 0$
- $\Phi = -\Psi$ (fine as we neglect quadrupoles)
- integrate radiation equations to get Θ_0 and Θ_1 equations

$$\Rightarrow \dot{\Theta} + ik\mu\Theta \simeq -\dot{\Phi} - ik\mu\Psi$$

$$\Theta_\ell = \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)$$

$$\Theta_0 = \int_{-1}^{1}\frac{d\mu}{2}\Theta(\mu) \quad [P_0(\mu) = 1]$$

$$\Theta_1 = i\int_{-1}^{1}d\mu\frac{\mu}{2}\Theta(\mu) \quad [P_1(\mu) = \mu]$$

$$\Theta_2 = \int_{-1}^{1}d\mu\left(\frac{1-3\mu^2}{4}\right)\Theta(\mu) \quad \left[P_2(\mu) = \frac{3\mu^2-1}{2}\right]$$

$$\times\int\frac{d\mu}{2} \Rightarrow \frac{d\Theta}{d\eta} + ik\mu\Theta = -\frac{d\Phi}{d\eta} + ik\mu\Phi$$

$$\Rightarrow \int_{-1}^{+1}\frac{d\mu}{2}\frac{d\Theta}{d\eta} + ik\underbrace{\int_{-1}^{1}\frac{d\mu}{2}\mu\Theta}_{\Theta_1} = -\int_{-1}^{1}\frac{d\mu}{2}\frac{d\Phi}{d\eta} + ik\Phi\int_{-1}^{1}\frac{d\mu}{2}\mu$$

$$\Rightarrow \frac{d}{d\eta}\underbrace{\int_{-1}^{1}\frac{d\mu}{2}\Theta}_{\Theta_0} + k\Theta_1 = -\frac{d\Phi}{d\eta}\underbrace{\int_{-1}^{1}\frac{d\mu}{2}}_{1} + ik\Phi\cancelto{0}{\left.\frac{\mu^2}{4}\right|_{-1}^{1}}$$

$$\Rightarrow \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \quad \text{monopole equation}$$

$$\frac{d\Theta}{d\eta} + ik\mu\Theta = -\frac{d\Phi}{d\eta} + ik\mu\Phi \qquad \times \int \frac{d\mu}{2}\mu$$

$$\int_{-1}^{1}\frac{d\mu}{2}\mu\frac{d\Theta}{d\eta} + ik\int_{-1}^{1}\frac{d\mu}{2}\mu^2\Theta = -\int_{-1}^{1}\frac{d\mu}{2}\mu\frac{d\Phi}{d\eta} + ik\Phi\int_{-1}^{1}\frac{d\mu}{2}\mu^2$$

$$\Rightarrow \frac{d}{d\eta}\underbrace{\int_{-1}^{1}\frac{d\mu}{2}\mu\Theta}_{-i\Theta_1} + ik\int_{-1}^{1}\frac{d\mu}{2}\mu^2\Theta = -\frac{d\Phi}{d\eta}\cancel{\underbrace{\int_{-1}^{1}d\mu\frac{\mu}{2}}_{=\frac{\mu^2}{4}\Big|_{-1}^{1}=0}} + ik\Phi\frac{\mu^3}{6}\Big|_{-1}^{1}$$

$$\Rightarrow -i\frac{d\Theta_1}{d\eta} + ik\int_{-1}^{1}\frac{d\mu}{2}\mu^2\Theta = \frac{ik\Phi}{3} \qquad \times i$$

$$\Rightarrow \dot{\Theta}_1 - k\int d\mu\frac{\mu^2}{2}\Theta = -\frac{k\Phi}{3}$$

Evaluate approximately by demanding $\Theta_2 = 0$

$$\Theta_2 = \int_{-1}^{1}d\mu\left(\frac{1-3\mu^2}{4}\right)\Theta = \int_{-1}^{1}\frac{d\mu}{4}\Theta - \frac{3}{4}\int_{-1}^{1}d\mu\,\mu^2\Theta = \frac{1}{2}\underbrace{\int_{-1}^{1}\frac{d\mu}{2}\Theta}_{\frac{1}{2}\Theta_0} - \frac{3}{2}\underbrace{\int_{-1}^{1}d\mu\frac{\mu^2}{2}\Theta}_{\text{unknown}} \approx 0$$

$$\Rightarrow \frac{1}{2}\Theta_0 - \frac{3}{2}\int_{-1}^{1}d\mu\frac{\mu^2}{2}\Theta \approx 0 \Rightarrow \int_{-1}^{1}d\mu\frac{\mu^2}{2}\Theta \approx \frac{\cancel{2}}{3}\frac{1}{\cancel{2}}\Theta_0 = \frac{\Theta_0}{3}$$

$$\Rightarrow \dot{\Theta}_1 - \frac{k\Theta_0}{3} = -\frac{k\Phi}{3}$$

Putting everything together our Boltzmann equations simplify to

$$\dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi}$$

$$\dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi$$

$$\dot{\delta} + ikv = -3\dot{\Phi}$$

$$\dot{v} + \frac{\dot{a}}{a}v = ik\Phi \qquad (\Phi = -\Psi)$$

We also need equation for Φ

Two options (are redundant since we set $\Phi=-\Psi$):

- Time-time equation

$$k^2\Phi + 3\frac{\dot a}{a}\left(\dot\Phi - \Psi\frac{\dot a}{a}\right) = 4\pi G a^2\left(\rho_{dm}\delta + 4\rho_r\Theta_{r,0}\right)$$ [neglected baryons]

$$\Downarrow \Phi=-\Psi$$

$$k^2\Phi + 3\frac{\dot a}{a}\left(\dot\Phi + \frac{\dot a}{a}\Phi\right) = 4\pi G a^2\left(\rho_{dm}\delta + 4\rho_r\Theta_{r,0}\right)$$

- Algebraic equation (exercise)

$$k^2\Phi = 4\pi G a^2\left[\rho_{dm}\delta + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}\left(i\rho_{dm}v + 4\rho_r\Theta_{r,1}\right)\right]$$

Full system of 5 equations for $\delta, v, \Theta_{r,0}, \Theta_{r,1}, \Phi$

$$\dot\Theta_{r,0} + k\Theta_{r,1} = -\dot\Phi$$

$$\dot\Theta_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi$$

$$\dot\delta + ikv = -3\dot\Phi$$

$$\dot v + \frac{\dot a}{a}v = ik\Phi$$

$$\left\{\begin{array}{l} k^2\Phi + 3\frac{\dot a}{a}\left(\dot\Phi + \frac{\dot a}{a}\Phi\right) = 4\pi G a^2\left(\rho_{dm}\delta + 4\rho_r\Theta_{r,0}\right) \\ \text{OR} \\ k^2\Phi = 4\pi G a^2\left[\rho_{dm}\delta + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}\left(i\rho_{dm}v + 4\rho_r\Theta_{r,1}\right)\right] \end{array}\right.$$

which one is more useful depends on the context

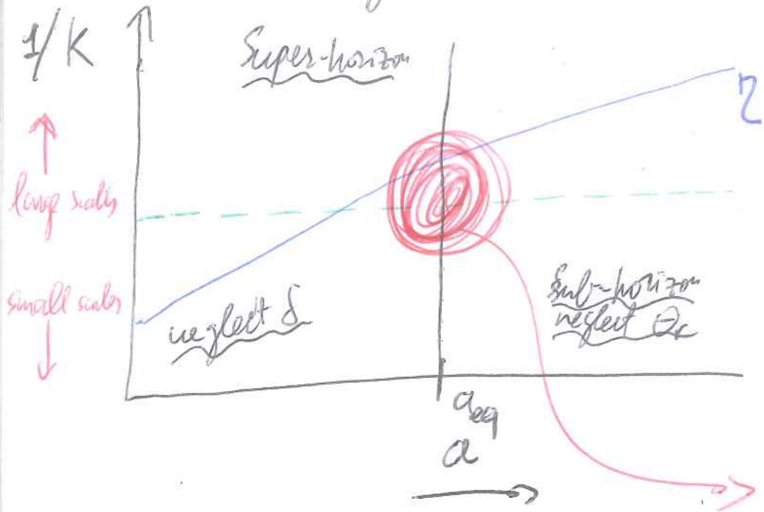
~~Analytic solutions for~~

$\hookrightarrow$ Very easy to solve numerically

Analytical solutions for δ are hard, exist only in certain limits (super- or sub-horizon, early- or late-time)

In general no analytic treatment for horizon crossing around equality

Solution strategy and where approximations exist



a given mode has constant comoving wavenumber, enters horizon when $k\eta \sim 1$

~~Solutions~~ analytical approximations exist everywhere except here, modes crossing horizon around equality

Large scales: • early times super-horizon, drop terms proportional to k
• late times neglect radiation ~~Θ_r~~, sub-horizon
• match super-horizon and sub-horizon solutions as both have $\Phi \sim$ const (crosses horizon during MD)

Small scales: • neglect matter perturbations at horizon crossing (since it is deep RD)
• late times neglect radiation, ~~Φ const~~
• match super-horizon and sub-horizon solutions

Intermediate scales which cross horizon around matter-radiation equality: no analytical solution, but transfer function is smooth, can spline large- and small-scale approximations

Now let's look at large scales first, starting super-horizon then through horizon crossing

Large scales (small k, enter horizon before equality!) $\rightarrow k \lesssim 0.01\, h\, \mathrm{Mpc}^{-1}$

Analytical solutions for Φ through $R \rightarrow M$ transition and then horizon crossing (note crossing deep during MD)