

Full set of equations (are algebraic Einstein equations)

$$\dot{\Theta}_{r,0} + K\Theta_{r,1} = -\dot{\Phi}$$

$$\dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi$$

$$\dot{\delta} + ikv = -3\dot{\Phi}$$

$$\dot{\Phi} + \frac{\dot{a}}{a}v = ik\Phi$$

$$K^2\Phi = 4\pi G a^2 \left[\rho_m \delta + 4p_r \Theta_{r,0} + \frac{3aH}{K} (i\rho_m v + 4p_r \Theta_{r,1}) \right]$$

Start considering only Θ_0, Θ_1, Φ

radiation and potential, drop all matter

$$\begin{cases} \dot{\Theta}_{r,0} + K\Theta_{r,1} = -\dot{\Phi} \\ \dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi \end{cases}$$

$$K^2\Phi = 4\pi G a^2 \left[\cancel{\rho_m \delta} + 4p_r \Theta_{r,0} + \frac{3aH}{K} (\cancel{i\rho_m v} + 4p_r \Theta_{r,1}) \right]$$

$$\Rightarrow K^2\Phi = 4\pi G a^2 \left(4p_r \Theta_{r,0} + \frac{12aH}{K} p_r \Theta_{r,1} \right) =$$

$$= 16\pi G p_r a^2 \left(\Theta_{r,0} + \frac{3aH}{K} \Theta_{r,1} \right) = 6a^2 H^2 \left(\Theta_{r,0} + \frac{3aH}{K} \Theta_{r,1} \right)$$

$$\uparrow H^2 \approx \frac{8\pi G}{3} \rho_r \Rightarrow 16\pi G p_r \approx 6H^2$$

$$\Rightarrow \Phi \approx \frac{6a^2 H^2}{K^2} \left(\Theta_{r,0} + \frac{3aH}{K} \Theta_{r,1} \right)$$

Useful relations during RD

$$aH = \frac{1}{\eta} \rightarrow \frac{d(aH)}{d\eta} = -\frac{1}{\eta^2} = -a^2 H^2$$

$$\frac{d[(aH)^{-1}]}{d\eta} = 1$$

$$\frac{d[(aH)^2]}{d\eta} = 2aH \frac{d(aH)}{d\eta} = -2a^3 H^3$$

$$\frac{d[(aH)^{-2}]}{d\eta} = \frac{d}{d\eta} \eta^2 = 2\eta = \frac{2}{aH}$$

$$\Phi = \frac{6a^2 H^2}{k^2} \left(\Theta_{r0} + \frac{3aH}{k} \Theta_{r1} \right) \quad \text{use it to eliminate } \Theta_{r0}$$

let me drop "r" for the moment

$$\text{in a) } \begin{cases} \dot{\Theta}_{r10} + k\Theta_{r1} = -\dot{\Phi} \\ \dot{\Theta}_{r11} - \frac{k}{3}\Theta_{r10} = -\frac{k}{3}\Phi \end{cases}$$

$$\Rightarrow \Theta_0 = \frac{k^2 \Phi}{6a^2 H^2} - \frac{3aH}{k} \Theta_1 = \frac{k^2 \eta^2}{6} \Phi - \frac{3}{k\eta} \Theta_1$$

$$\dot{\Theta}_0 = \frac{k^2 \eta}{3} \dot{\Phi} + \frac{k^2 \eta^2}{6} \dot{\Phi} + \frac{3}{k\eta^2} \dot{\Theta}_1 - \frac{3}{k\eta} \dot{\Theta}_1$$

$$\text{a) } \Rightarrow \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \quad \text{becomes}$$

$$\frac{k^2 \eta}{3} \dot{\Phi} + \frac{k^2 \eta^2}{6} \dot{\Phi} + \frac{3}{k\eta^2} \dot{\Theta}_1 - \frac{3}{k\eta} \dot{\Theta}_1 + k\Theta_1 = -\dot{\Phi}$$

$$\Rightarrow \left[-\frac{3}{k\eta} \dot{\Theta}_1 + k\Theta_1 \left(1 + \frac{3}{k^2 \eta^2} \right) = -\dot{\Phi} \left(1 + \frac{k^2 \eta^2}{6} \right) - \Phi \frac{k^2 \eta}{3} \right]$$

$$\text{b) } \Rightarrow \dot{\Theta}_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\Phi \quad \text{becomes}$$

$$\dot{\Theta}_1 - \frac{k}{3} \left(\frac{k^2 \eta^2}{6} \Phi - \frac{3}{k\eta} \Theta_1 \right) = -\frac{k}{3}\Phi \quad \Rightarrow \left[\dot{\Theta}_1 + \frac{1}{\eta} \Theta_1 = -\frac{k}{3}\Phi \left(1 - \frac{k^2 \eta^2}{6} \right) \right]$$

Now we have two 1st order equations for Φ, Θ_1

$$\begin{cases} -\frac{3}{k\eta} \dot{\Theta}_1 + k\Theta_1 \left(1 + \frac{3}{k^2 \eta^2} \right) = -\dot{\Phi} \left(1 + \frac{k^2 \eta^2}{6} \right) - \Phi \frac{k^2 \eta}{3} \\ \dot{\Theta}_1 + \frac{1}{\eta} \Theta_1 = -\frac{k}{3}\Phi \left(1 - \frac{k^2 \eta^2}{6} \right) \end{cases} \quad \longrightarrow$$

Turn it into 2nd order equation for Φ

by repeatedly differentiating

$$-\frac{3}{k\eta} \dot{\theta}_1 + k\theta_1 \left(1 + \frac{3}{k^2\eta^2}\right) = -\ddot{\phi} \left(1 + \frac{k^2\eta^2}{6}\right) - \frac{\phi k^2\eta}{3}$$

$$\Downarrow \dot{\theta}_1 + \frac{\theta_1}{\eta} = -\frac{k}{3}\phi \left(1 - \frac{k^2\eta^2}{6}\right) \Rightarrow \dot{\theta}_1 = -\frac{k\phi}{3} \left(1 - \frac{k^2\eta^2}{6}\right) - \frac{\theta_1}{\eta}$$

$$-\frac{3}{k\eta} \left[-\frac{\theta_1}{\eta} - \frac{k}{3}\phi \left(1 - \frac{k^2\eta^2}{6}\right) \right] + k\theta_1 \left(1 + \frac{3}{k^2\eta^2}\right) + \ddot{\phi} \left(1 + \frac{k^2\eta^2}{6}\right) + \frac{k^2\eta}{3}\phi = 0$$

$$\Rightarrow \frac{3}{k\eta^2} \theta_1 + \frac{\phi}{\eta} \left(1 - \frac{k^2\eta^2}{6}\right) + k\theta_1 + \frac{3}{k\eta^2} \theta_1 + \ddot{\phi} \left(1 + \frac{k^2\eta^2}{6}\right) + \frac{k^2\eta}{3}\phi =$$

$$= \frac{3}{k\eta^2} \theta_1 + \frac{\phi}{\eta} - \frac{k^2\eta}{6}\phi + k\theta_1 + \frac{3}{k\eta^2} \theta_1 + \ddot{\phi} + \frac{k^2\eta^2}{6}\ddot{\phi} + \frac{k^2\eta}{3}\phi =$$

$$= \ddot{\phi} + \frac{k^2\eta^2}{6}\ddot{\phi} + \frac{\phi}{\eta} + \frac{k^2\eta}{6}\phi + k\theta_1 + \frac{6}{k\eta^2} \theta_1 =$$

$$= \ddot{\phi} \left(1 + \frac{k^2\eta^2}{6}\right) + \frac{\phi}{\eta} \left(1 + \frac{k^2\eta^2}{6}\right) + k\theta_1 \left(1 + \frac{6}{k^2\eta^2}\right) = 0$$

$$\Rightarrow \ddot{\phi} + \frac{\phi}{\eta} = -\frac{\left(1 + \frac{6}{k^2\eta^2}\right) k\theta_1}{\left(1 + \frac{k^2\eta^2}{6}\right)} = -\frac{k^2\eta^2 + 6}{k^2\eta^2} \frac{6}{6 + k^2\eta^2} k\theta_1 = -\frac{6}{k\eta^2} \theta_1$$

$$\Rightarrow \boxed{\ddot{\phi} + \frac{\phi}{\eta} = -\frac{6}{k\eta^2} \theta_1} \rightarrow \begin{cases} \theta_1 = -\frac{k\eta^2}{6} \left(\ddot{\phi} + \frac{\phi}{\eta}\right) \\ \dot{\theta}_1 = -\frac{k}{3}\phi \left(1 - \frac{k^2\eta^2}{6}\right) - \frac{\theta_1}{\eta} = \end{cases}$$

Now differentiate and replace $\theta_1, \dot{\theta}_1$

$$= -\frac{k}{3}\phi \left(1 - \frac{k^2\eta^2}{6}\right) + \frac{k\eta}{6} \left(\ddot{\phi} + \frac{\phi}{\eta}\right)$$

$$\ddot{\phi} + \frac{\phi}{\eta} = -\frac{6}{k\eta^2} \theta_1 \Rightarrow \ddot{\phi} + \frac{\phi}{\eta} + \frac{6}{k\eta^2} \theta_1 = 0$$

~~Derivative~~ $\frac{d}{d\eta}$

Now take $\frac{d}{d\eta}$

$$\begin{aligned} \frac{d}{d\eta} \left(\dot{\Phi} + \frac{\Phi}{\eta} + \frac{6}{k\eta^2} \Theta \right) &= \ddot{\Phi} - \frac{\Phi}{\eta^2} + \frac{\dot{\Phi}}{\eta} - \frac{12}{k\eta^3} \Theta + \frac{6}{k\eta^2} \dot{\Theta} = \\ &= \ddot{\Phi} - \frac{\Phi}{\eta^2} + \frac{\dot{\Phi}}{\eta} - \frac{12}{k\eta^3} \left[-\frac{k\eta^2}{6} \left(\dot{\Phi} + \frac{\Phi}{\eta} \right) \right] + \frac{6}{k\eta^2} \left[-\frac{k}{3} \Phi \left(1 - \frac{k\eta^2}{6} \right) + \frac{k\eta}{6} \left(\dot{\Phi} + \frac{\Phi}{\eta} \right) \right] = \\ &= \ddot{\Phi} - \cancel{\frac{\Phi}{\eta^2}} + \frac{\dot{\Phi}}{\eta} + \frac{2}{\eta} \dot{\Phi} + \cancel{\frac{2}{\eta^2} \Phi} - \cancel{\frac{2}{\eta^2} \Phi} + \frac{k^2 \Phi}{3} + \frac{\dot{\Phi}}{\eta} + \cancel{\frac{\Phi}{\eta^2}} = \\ &= \ddot{\Phi} + \frac{4}{\eta} \dot{\Phi} + \frac{k^2 \Phi}{3} \end{aligned}$$

$$\Rightarrow \boxed{\ddot{\Phi} + \frac{4}{\eta} \dot{\Phi} + \frac{k^2 \Phi}{3} = 0}$$

Remember at very (very!) early times

$$\ddot{\Phi} \eta + 4 \dot{\Phi} = 0 \rightarrow \Phi = \eta^p \quad p=0, -3 \leftarrow \text{decaying mode we don't care about it}$$

$\Phi = \text{const mode excited!}$

→ Solve this with constraint/initial condition $\Phi = \text{const}$

Analytical solution by defining $u \equiv \Phi \eta \rightarrow \Phi = \frac{u}{\eta}, \dot{\Phi} = \frac{\dot{u}}{\eta} - \frac{u}{\eta^2}$

$$\ddot{\Phi} = \frac{\ddot{u}}{\eta} - \frac{2\dot{u}}{\eta^2} + \frac{2u}{\eta^3}$$

$$\ddot{\Phi} + \frac{4}{\eta} \dot{\Phi} + \frac{k^2}{3} \Phi = \frac{\ddot{u}}{\eta} - \frac{2\dot{u}}{\eta^2} + \frac{2u}{\eta^3} + \frac{4}{\eta^2} \dot{u} - \frac{4}{\eta^3} u + \frac{k^2}{3\eta} u = 0$$

$$\Rightarrow \ddot{u} + \frac{2}{\eta} \dot{u} + \left(\frac{k^2}{3} - \frac{2}{\eta^2} \right) u = 0$$

It's of the Bessel function form! Recall ^{spherical} Bessel functions

l th spherical Bessel function solves

$$\frac{d^2 J_l}{dx^2} + \frac{2}{x} \frac{dJ_l}{dx} + \left[1 - \frac{l(l+1)}{x^2} \right] J_l = 0$$

$l=0 \rightarrow \frac{d^2 J_0}{dx^2} + \frac{2}{x} \frac{dJ_0}{dx} + J_0 = 0 \quad J_0(x) = \frac{\sin(x)}{x}$
 $l=1 \rightarrow \frac{d^2 J_1}{dx^2} + \frac{2}{x} \frac{dJ_1}{dx} + \left(1 - \frac{2}{x^2} \right) J_1 = 0 \quad J_1(x) = \frac{\sin(x) - x \cos(x)}{x^2}$

Solutions: $J_1\left(\frac{k\eta}{\sqrt{3}}\right)$ and $Y_1\left(\frac{k\eta}{\sqrt{3}}\right)$
 ↑
 spherical Bessel function

Blows up as $\eta \rightarrow 0$
 ↑
 spherical Neumann function

$u = \Phi \eta \sim \eta$ at early times

$$\frac{\sin x - x \cos x}{x^2} \sim \frac{x}{3} + O(x^3)$$

So $u = C J_1\left(\frac{k\eta}{\sqrt{3}}\right) = C \left[\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3}) \cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^2} \right]$

$$\Phi = \frac{u}{\eta} = C \left[\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3}) \cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3} \right] \xrightarrow{\eta \rightarrow 0} \frac{C}{3} = \Phi_p$$

$$\Rightarrow C = 3\Phi_p$$

So full solution is

$$\Phi = 3\Phi_p \left[\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3}) \cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3} \right]$$

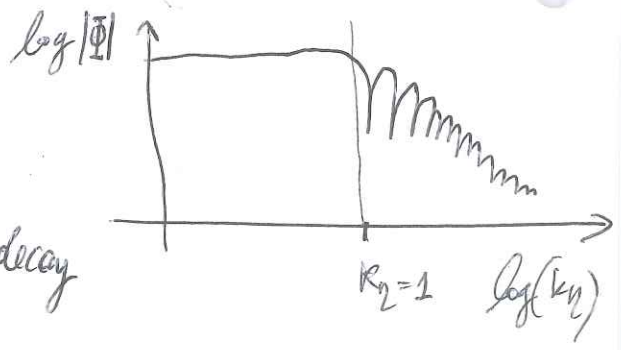
$k\eta \ll 1$ super-horizon

$\Phi \propto \text{const}$

$k\eta \gg 1$ sub-horizon

$\Phi \propto \text{oscillations} \times \text{decay}$

$k\eta \approx 1$ horizon crossing



So as soon as mode crosses horizon during RD the potential starts to decay, then oscillates

↑
 dilution of energy density due to expansion

↓
 due to radiation pressure

Heuristically

$$\Phi = \frac{6a^2 H^2}{k^2} \left(\theta_{r,0} + \frac{3aH}{k} \theta_{r,1} \right) \xrightarrow[\substack{\text{sub-horizon } k \gg aH \\ k \gg 1}]{} \Phi \sim \frac{\theta_0}{k^2}$$

This is the large k limit of the full solution

So far we have determined Φ , now determine δ

$$\begin{cases} \dot{\delta} + ikv = -3\dot{\Phi} \\ \dot{v} + \frac{\dot{a}}{a}v = ik\Phi \end{cases} \longrightarrow \text{Differentiate, then eliminate } v, \text{ then eliminate } \dot{v}$$

$$\frac{d}{d\eta} (\dot{\delta} + ikv) = \frac{d}{d\eta} (-3\dot{\Phi}) \Rightarrow \ddot{\delta} + ik\dot{v} = -3\ddot{\Phi}$$

$$v = -\frac{\dot{a}}{a}v + ik\Phi$$

$$= \ddot{\delta} + ik\left(-\frac{\dot{a}}{a}v + ik\Phi\right) = \ddot{\delta} - ik\frac{\dot{a}}{a}v - k^2\Phi =$$

$$= \ddot{\delta} - ik\frac{\dot{a}}{a}\left(\frac{3i\dot{\Phi}}{k} + \frac{i\dot{\delta}}{k}\right) - k^2\Phi = \ddot{\delta} + 3\frac{\dot{a}}{a}\dot{\Phi} + \frac{\dot{a}}{a}\dot{\delta} - k^2\Phi$$

$$v = -\frac{3\dot{\Phi}}{ik} - \frac{\dot{\delta}}{ik} = \frac{3i\dot{\Phi}}{k} + \frac{i\dot{\delta}}{k}$$

During RD

$$\frac{\dot{a}}{a} = \frac{1}{a} \frac{da}{d\eta} = a \left(\frac{1}{a} \frac{da}{dt} \right) = aH = \frac{1}{2}$$

$$\Rightarrow \ddot{\delta} + 3\frac{\dot{a}}{a}\dot{\Phi} + \frac{\dot{a}}{a}\dot{\delta} - k^2\Phi = -3\ddot{\Phi}$$

$$\Rightarrow \boxed{\ddot{\delta} + \frac{1}{\eta}\dot{\delta} = S(k, \eta)}$$

$$\text{where } S(k, \eta) = -3\ddot{\Phi} + k^2\Phi - \frac{3}{\eta}\dot{\Phi}$$

$$\ddot{\delta} + \frac{1}{\eta}\dot{\delta} = -3\ddot{\Phi} + k^2\Phi - \frac{3}{\eta}\dot{\Phi} \quad \text{equation for } \delta \text{ perturbations } \left. \begin{array}{l} \text{+ RD} \\ \text{- around horizon crossing} \end{array} \right\}$$

Homogeneous equation

$$\ddot{\delta} + \frac{1}{\eta}\dot{\delta} = 0 \begin{cases} \delta = \text{const} \\ \delta = \ln(\eta) [= \ln(a) \text{ during RD}] \end{cases}$$

$$y = \delta \quad y' + \frac{y}{\eta} = 0 \quad \frac{dy}{d\eta} = -\frac{y}{\eta} \quad \frac{dy}{y} = -\frac{d\eta}{\eta} \quad y \propto \frac{1}{\eta}$$

$\delta = \text{const}$, $\delta = \ln(\eta)$ solutions to homogeneous equation

$$s_1(\eta) \quad s_2(\eta)$$

Recall initial condition $\delta(k, \eta_i) = \text{const} = \frac{3}{2} \Phi_p$

General solution to 2nd order equation

~~$\delta(k, \eta) = C_1 s_1(\eta) + C_2 s_2(\eta) + \int_0^\eta d\eta' S(k, \eta') \frac{s_1(\eta) s_2(\eta') - s_1(\eta') s_2(\eta)}{s_1(\eta') s_2'(\eta') - s_1'(\eta') s_2(\eta')}$~~

$$\delta(k, \eta) = C_1 s_1(\eta) + C_2 s_2(\eta) + \int_0^\eta d\eta' S(k, \eta') \frac{s_1(\eta) s_2(\eta') - s_1(\eta') s_2(\eta)}{s_1(\eta') s_2'(\eta') - s_1'(\eta') s_2(\eta')}$$

$$G(\eta, \eta') = \frac{1 \times \ln(\eta') - 1 \times \ln(\eta)}{0 - 1/\eta'} = -\eta' [\ln(k\eta') - \ln(k\eta)]$$

$$\Rightarrow \delta(k, \eta) = C_1 + C_2 \ln(\eta) - \int_0^\eta d\eta' S(k, \eta') \eta' [\ln(k\eta') - \ln(k\eta)]$$

$$\xrightarrow{\eta \rightarrow 0} C_1 + C_2 \ln(\eta) = \frac{3}{2} \Phi_p \Rightarrow C_2 = 0$$

$$C_1 = \frac{3\Phi_p}{2}$$

How does $S(k, \eta')$ behave?

When mode enters horizon $k\eta \approx 1$, Φ decays. For $k\eta < 1$ integral is small

$$S(k, \eta) = -3\ddot{\Phi} + k^2\Phi - \frac{3}{\eta}\dot{\Phi} \longrightarrow 0$$

Dominant contribution to integral $\int d\eta' S(k, \eta') G(\eta, \eta')$ when $k\eta \approx 1$

$$\int d\eta' S(k, \eta') G(\eta, \eta') \propto \int d\eta' S(\eta') \eta' \ln(k\eta') \approx \int d\eta' S(\eta') \ln(k\eta)$$

asymptote to constant $\propto \ln(k\eta)$

So after mode entered horizon

$$\delta(k, \eta) \propto \ln(k\eta) = A\Phi_p \ln(Bk\eta) = \underbrace{A\Phi_p \ln(B)}_{\text{constant}} + \underbrace{A\Phi_p \ln(k\eta)}_{\text{logarithmically growing mode}}$$

$$\delta(k, \eta) = C_1 + C_2 \ln(\eta) - \int_0^{\eta} d\eta' S(k, \eta') \eta' (\ln(k\eta') - \ln(k\eta))$$

$$= A \Phi_p \ln(B) + A \Phi_p \ln(k\eta)$$

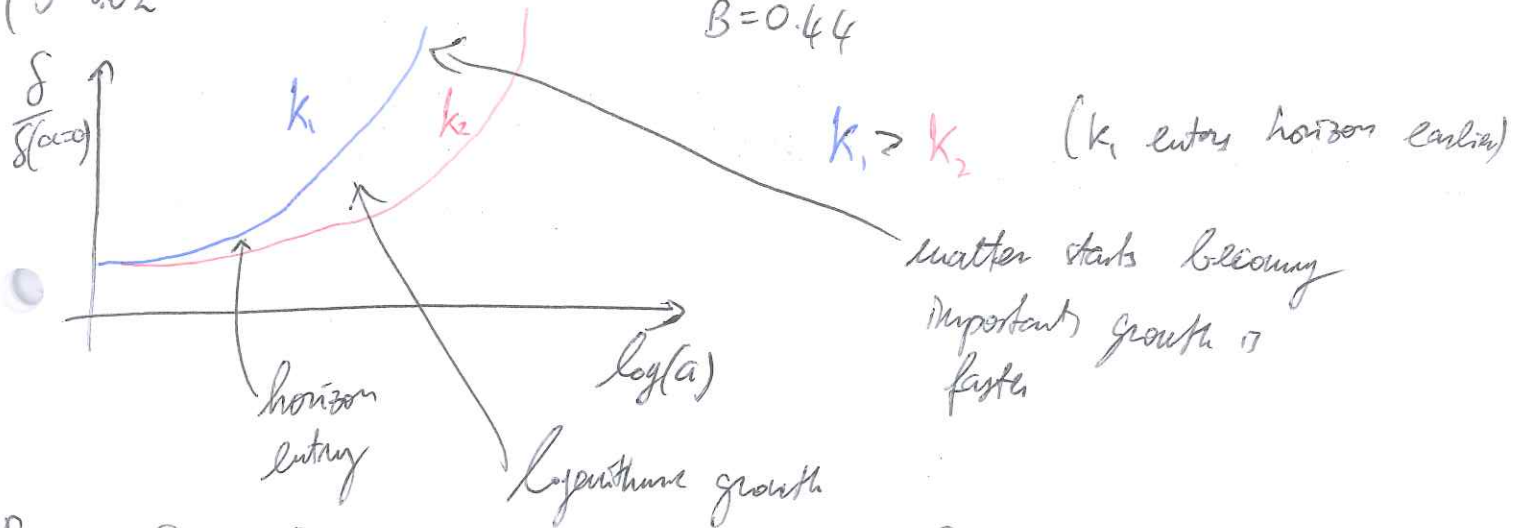
$$\rightarrow A \Phi_p \ln(B) = C_1 - \int_0^{\infty} d\eta' S(k, \eta') \eta' \ln(k\eta')$$

$$A \Phi_p = \int_0^{\infty} d\eta' S(k, \eta') \eta'$$

Upper limits of integration at ∞ since integrals asymptote to constant at large η

Knowing $\Phi \propto J_1(k\eta)$ we can do the integrals numerically

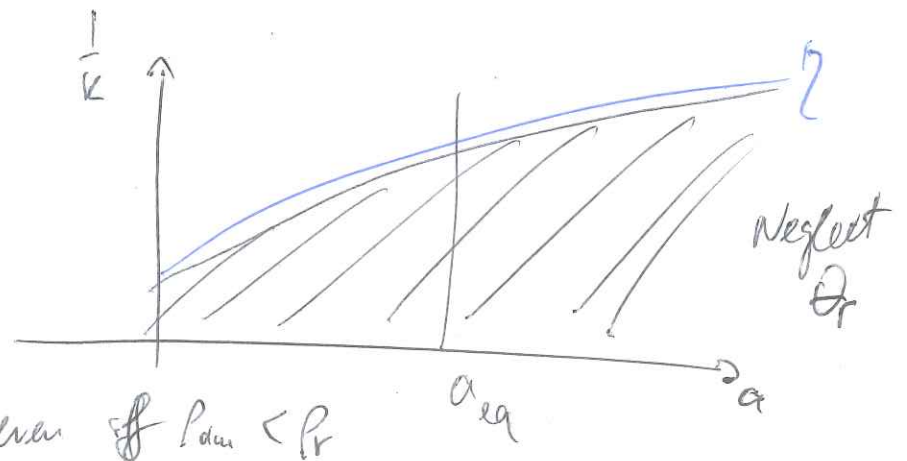
$\begin{cases} A=9.0 \\ B=0.62 \end{cases} \xrightarrow{\text{integrating exact solution}} \begin{cases} A=9.6 \\ B=0.44 \end{cases}$



Recap: $\delta \propto \ln(\eta) \propto \ln(a)$ during RD $\delta \propto a$ during RD

Sub-horizon solution

Pressure also suppresses growth of θ_r (while δ grows as $\delta \propto \ln \eta$)



At some point $\rho_{dm} \delta \gtrsim \rho_r \theta_{r,0}$ even if $\rho_{dm} < \rho_r$

Once this happens, Φ and δ evolve together independent of radiation, so we can solve for Φ and δ matching on to logarithmic growing solution set up while Φ was decaying

$$\delta(k, z) = A \Phi_p \ln(Bkz)$$

Full set of equations (use algebraic Einstein equations)

$$\begin{cases} \dot{\Theta}_{r,0} + k \Theta_{r,1} = -\dot{\Phi} \\ \dot{\Theta}_{r,1} - \frac{k}{3} \Theta_{r,0} = -\frac{k}{3} \Phi \\ \dot{\delta} + i k v = -3\dot{\Phi} \\ \dot{v} + \frac{\dot{a}}{a} v = i k \Phi \\ k^2 \Phi = 4\pi G a^2 \left[\rho_{dm} \delta + 4\rho_r \Theta_{r,0} + \frac{3aH}{k} (i \rho_{dm} v + 4\rho_r \Theta_{r,1}) \right] \end{cases} \quad \text{neglect radiation}$$

$$\rightarrow \begin{cases} \dot{\delta} + i k v = -3\dot{\Phi} \\ \dot{v} + \frac{\dot{a}}{a} v = i k \Phi \\ k^2 \Phi \approx 4\pi G a^2 \left(\rho_{dm} \delta + \frac{3i a H}{k} \rho_{dm} v \right) \end{cases} \quad \begin{array}{l} \text{As usual we want to} \\ \text{reduce these 3 equations} \\ \text{to 1 2nd order equation} \\ \text{for } \delta \end{array}$$

We need to follow δ through equality, so again use variable y

Recall $y \equiv \frac{a}{a_{eq}}$ ignoring baryons so $\frac{\rho_r}{\rho_{dm}} \approx \frac{a_{eq}}{a} \approx \frac{1}{y}$

$$\frac{d}{dz} = \dots = a k y \frac{d}{dy} = a k y' \quad \text{so } \cdot \rightarrow a k y'$$

↑ done earlier

$$\dot{\delta} + i k v = -3\dot{\Phi} \Rightarrow a k y \delta' + i k v = -3 a k y \Phi' \Rightarrow \delta' + \frac{i k v}{a k y} = -3\Phi'$$

$$\dot{v} + \frac{\dot{a}}{a} v = i k \Phi \Rightarrow a k y v' + \frac{a k y}{a} v' = i k \Phi$$

$$\frac{aHv}{a} a' v = Hya'v =$$

$$\left\{ \begin{aligned} a' &= \frac{da}{dy} = \left| y = \frac{a}{a_{eq}} \rightarrow dy = \frac{1}{a_{eq}} da \rightarrow \frac{d}{dy} = a_{eq} \frac{d}{da} \right. \\ &= a_{eq} \frac{da}{da} = a_{eq} \end{aligned} \right.$$

$$= Hya_{eq} v =$$

$$= H \frac{a}{a_{eq}} a_{eq} v = H a v$$

$$\Rightarrow aHv' + aHv = iK\Phi \implies v' + \frac{v}{y} = \frac{iK\Phi}{aHv}$$

$$K^2 \Phi \approx 4\pi b a^2 \left(\rho_{dm} \delta + \frac{3iaH}{K} \rho_{dm} v \right) \stackrel{\text{Sub-horizon modes}}{aH \ll 1} \approx 4\pi b a^2 \rho_{dm} \delta =$$

$$\text{Recall } y \approx \frac{\rho_{dm}}{\rho} \Rightarrow y+1 \approx \frac{\rho}{\rho_r} \Rightarrow \frac{y}{y+1} \rho \approx \rho_{dm}$$

$$= 4\pi b a^2 \rho \frac{y}{y+1} \delta = 4\pi b a^2 \frac{3H^2}{8\pi G} \frac{y}{y+1} = \frac{3}{2} a^2 H^2 \frac{y}{y+1} \delta$$

\uparrow $3H^2 = \frac{8\pi G}{3} \rho$

So final set of equations

$$\begin{cases} \delta' + \frac{iKv}{aHy} = -3\Phi' \\ v' + \frac{v}{y} = \frac{iK\Phi}{aHy} \\ K^2 \Phi = \frac{3y}{2(y+1)} a^2 H^2 \delta \end{cases}$$

Now let's turn these into a 2nd order equation for δ

$$\frac{d}{dy} \left(\delta' + \frac{iKv}{aHy} \right) = \frac{d}{dy} (-3\Phi')$$

$$\text{Useful: } \frac{d}{dy} \left(\frac{1}{aHy} \right) = -\frac{1}{2aHy(1+y)}$$

Exercise!

$$\frac{d}{dy} \left(\delta' + \frac{ikv}{aHy} \right) = -3\Phi''$$

$$\frac{d}{dy} \left(\delta' + \frac{ikv}{aHy} \right) = \delta'' + \frac{ik}{aHy} v' + ikv \frac{d}{dy} \left(\frac{1}{aHy} \right) = \delta'' + \frac{ik}{aHy} \left(\frac{ik\Phi}{aHy} - \frac{v}{y} \right) + ikv \left(-\frac{1}{2aHy(1+y)} \right)$$

$$= \delta'' - \frac{k^2\Phi}{a^2H^2y^2} - \frac{ik}{aHy^2} v - \frac{ikv}{2aHy(1+y)} = \delta'' - \frac{k^2\Phi}{a^2H^2y^2} - ikv \left(\frac{1}{aHy^2} + \frac{1}{2aHy(1+y)} \right) =$$

$$\frac{2(1+y) + y}{2aHy^2(1+y)} = \frac{2+3y}{2aHy^2(1+y)}$$

$$= \delta'' - \frac{k^2\Phi}{a^2H^2y^2} - \frac{ik(2+3y)v}{2aHy^2(1+y)} = -3\Phi''$$

$$\Rightarrow \delta'' - \frac{ik(2+3y)v}{2aHy^2(1+y)} = -3\Phi'' + \frac{k^2\Phi}{a^2H^2y^2} \approx \frac{k^2\Phi}{a^2H^2y^2} = \frac{3\delta}{2y(y+1)} \quad *$$

$k \gg aH$

$$k^2\Phi = \frac{3y}{2(y+1)} a^2H^2\delta$$

$$\frac{ik(2+3y)v}{2aHy^2(1+y)} = -\frac{ik(2+3y)}{2aHy^2(1+y)} \frac{iaHy}{k} (3\Phi' + \delta) \approx -\frac{ik(2+3y)}{2aHy^2(1+y)} \frac{iaHy}{k} \delta' =$$

$$\delta' + \frac{ikv}{aHy} = -3\Phi' \Rightarrow v = \frac{iaHy}{k} (3\Phi' + \delta) \approx \frac{iaHy}{k} \delta' \quad \text{since } \Phi \ll \delta \text{ on sub-horizon scales}$$

$$= \frac{2+3y}{2y(1+y)} \delta'$$

$$* \quad \delta'' + \frac{2+3y}{2y(1+y)} \delta' - \frac{3}{2y(y+1)} \delta = 0$$

Mestard's equation for sub-horizon CDM perturbations with negligible radiation perturbations for $z \ll z_{eq}$

$$\delta(y) = \delta \left(\frac{a}{a_{eq}} \right)$$

To see how δ grows we need 2 independent solutions of the Messerao equation, then match on to the logarithmic growth

$$\delta'' + \frac{2+3y}{2y(y+1)} \delta' - \frac{3}{2y(y+1)} \delta = 0$$

We know that deep during MD $\delta \propto a \, dy$, so one solution is a polynomial of order 1 in $y \rightarrow \delta \propto y \rightarrow \delta'' = 0$

For this ~~mode~~ growing mode, as long as $a \lesssim 0.1$ (afterwards DE is dominant so we need growth function)

$\delta = D_1 \propto y$

$$\cancel{D_1}'' + \frac{2+3y}{2y(y+1)} \cancel{D_1}' - \frac{3}{2y(y+1)} \cancel{D_1} = 0 \rightarrow \frac{2+3y}{2y(y+1)} D_1' = \frac{3}{2y(y+1)} D_1$$

$$\Rightarrow \boxed{\frac{D_1'}{D_1} = \frac{3}{2+3y}} \quad \text{for the Messerao growing mode}$$

$$\frac{1}{D_1} \frac{dD_1}{dy} = \frac{3}{2+3y} \Rightarrow \int \frac{dD_1}{D_1} = \int \frac{3}{2+3y} dy \Rightarrow \ln D_1 = \ln(2+3y) + \text{const}$$

$$\Rightarrow D_1(y) = 2+3y \quad \text{growing mode} \quad \delta(y) = 2+3y \propto y$$

To find the second solution consider $u = \frac{\delta}{D_1} = \frac{\delta}{y+\frac{2}{3}} = \frac{3}{2+3y} \delta$

$$\rightarrow \delta = \frac{2+3y}{3} u(y)$$

$$\delta' = u(y) + \frac{2+3y}{3} u'(y)$$

$$\delta'' = 2u'(y) + \frac{2+3y}{3} u''(y)$$

$$\delta'' + \frac{2+3y}{2y(y+1)} \delta' - \frac{3}{2y(y+1)} \delta = 2u' + \frac{2+3y}{3} u'' + \left(\frac{2+3y}{2y(y+1)} \right) \left[u + \frac{2+3y}{3} u' \right]$$

$$- \frac{3}{2y(y+1)} \frac{2+3y}{3} u =$$

$$= \frac{2+3y}{3} u'' + \frac{(2+3y)^2}{6y(y+1)} u' + 2u + \frac{2+3y}{2y(y+1)} u - \frac{2+3y}{2y(y+1)} u =$$

$$= \frac{2+3y}{3} u'' + \frac{4+9y^2+12y+12y^2+12y}{6y(y+1)} = \frac{2+3y}{3} u'' + \frac{21y^2+24y+4}{6y(y+1)} u' = 0$$

Multiply by $\frac{3}{2}$

$$\Rightarrow \left(\left(1 + \frac{3y}{2} \right) u'' + \frac{\left(\frac{21}{4} y^2 + 6y + 1 \right)}{y(y+1)} u' \right) = 0$$

No term proportional to u' , so it is really a 1st order equation for u'

Call $x = u'$

$$\left(1 + \frac{3y}{2} \right) x' + \frac{\left(\frac{21}{4} y^2 + 6y + 1 \right)}{y(y+1)} x = 0 \quad \rightarrow \quad \frac{dx}{x} = - \frac{\frac{21}{4} y^2 + 6y + 1}{\left(1 + \frac{3y}{2} \right) y(y+1)} dy$$

$$\int \frac{dx}{x} = - \int dy \frac{\frac{21}{4} y^2 + 6y + 1}{\left(1 + \frac{3y}{2} \right) y(y+1)} = - \ln y - \frac{1}{2} \ln(4y) - 2 \ln(2+3y)$$

$$\Rightarrow x = u' \propto \left(y + \frac{2}{3} \right)^{-2} y^{-1} (y+1)^{-1/2}$$

$$u(y) = \int dy u'(y) = \int dy \left[\left(y + \frac{2}{3} \right)^{-2} y^{-1} (y+1)^{-1/2} \right]$$

Integrating one gets

$$\bullet \quad u \propto \frac{\sqrt{1+y}}{D_1(y)} - \operatorname{arctanh}(\sqrt{1+y}) = \frac{D_2(y)}{D_1(y)}$$

$$\Rightarrow D_2(y) = D_1(y) \ln \left[\frac{\sqrt{1+y} + 1}{\sqrt{1+y} - 1} \right] - 2\sqrt{1+y}$$

Late times $a \gg a_{eq}, y \gg 1$

$$D_1(y) = y + \frac{2}{3} \propto y \quad D_2(y) \propto y^{-3/2} \text{ decaying mode}$$

• General solution to Nečas equation

$$\delta(k, y) = C_1 D_1(y) + C_2 D_2(y) \quad \text{for } y > \frac{y}{y_H} \rightarrow \text{when the mode enters the horizon}$$

C_1 and C_2 determined matching to earlier logarithmic solution valid for $y_H \ll y \ll 1$ (modes enter horizon before equality)

Match two solutions and their derivatives

$$\bullet \quad \delta(k, y) = A \Phi_p \ln(B k y) \simeq A \Phi_p \ln\left(B \frac{y}{y_H}\right) \quad \text{valid for } y \ll 1$$

$$\bullet \quad \delta(k, y) = C_1 D_1(y) + C_2 D_2(y)$$

at matching time $y_H \ll y \ll 1$

$$A \Phi_p \ln\left(B \frac{y_H}{y_H}\right) = C_1 D_1(y_H) + C_2 D_2(y_H)$$

$$\frac{A \Phi_p}{y_H} = C_1 D_1'(y_H) + C_2 D_2'(y_H)$$

• Summing up:

\sim logarithmic growth + $\propto a$ growth