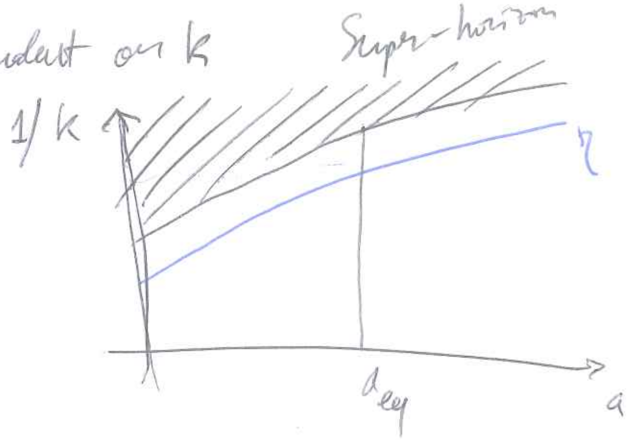
Super-horizon solution

- $k\eta \ll 1 \longrightarrow$ Neglect all terms dependent on k



Choose Einstein equation with derivatives (else $\sim \frac{1}{k}$ terms)

$$\begin{cases} \dot\Theta_{r,0} + \cancel{k\Theta_{r,1}} = -\dot\Phi \\ \dot\Theta_{r,1} \cancel{- \frac{k}{3}\Theta_{r,0}} = \cancel{-\frac{k}{3}\Phi} \\ \dot\delta + \cancel{ikv} = -3\dot\Phi \\ \dot v + \frac{\dot a}{a} v = \cancel{ik\Phi} \\ \cancel{k^2\Phi} + 3\frac{\dot a}{a}\left(\dot\Phi + \frac{\dot a}{a}\Phi\right) = 4\pi G a^2\left(\rho_{dm}\delta + 4\rho_r\Theta_{r,0}\right) \end{cases}
\longrightarrow \quad \begin{aligned} &\dot\Theta_{r,1} = 0 \\ &\dot v + \frac{\dot a}{a} v = 0 \end{aligned}$$

decouple

We go from 5 to 3 effective variables

$$\Rightarrow \begin{cases} \dot\Theta_{r,0} \approx -\dot\Phi \\ \dot\delta \approx -3\dot\Phi \\ 3\frac{\dot a}{a}\left(\dot\Phi + \frac{\dot a}{a}\Phi\right) \approx 4\pi G a^2\left(\rho_{dm}\delta + 4\rho_r\Theta_{r,0}\right) \end{cases} \longrightarrow \dot\delta - 3\dot\Theta_{r,0} = 0 \Rightarrow \delta - 3\Theta_{r,0} = \text{const}$$

- $\delta - 3\Theta_{r,0} = \text{const}$ but initial conditions tell us $\delta(k,\eta_i) = 3\Theta_0(k,\eta_i)$

$\Rightarrow \text{const} = 0$

So we can set $\delta = 3\Theta_{r,0} \longrightarrow \Theta_{r,0} = \frac{\delta}{3}$

$$3\frac{\dot a}{a}\left(\dot\Phi + \frac{\dot a}{a}\Phi\right) = 4\pi G a^2\left(\rho_{dm}\delta + 4\rho_r\Theta_{r,0}\right)$$

$\Downarrow$ Define $y \equiv \frac{a}{a_{eq}} \Rightarrow \rho_{dm,0} a_{eq}^{-3} = \rho_{r,0} a_{eq}^{-4} \Rightarrow a_{eq} = \frac{\rho_{r,0}}{\rho_{dm,0}}$ $\quad \frac{\rho_r}{\rho_{dm}} = \frac{\rho_{r,0} a^{-4}}{\rho_{dm,0} a^{-3}} = \frac{\rho_{r0}}{\rho_{dm0}}\frac{1}{a} = \frac{a_{eq}}{a}$

$$3\frac{\dot a}{a}\left(\dot\Phi + \frac{\dot a}{a}\Phi\right) = 4\pi G a^2\left(\rho_{dm}\delta + \frac{4}{3}\rho_r\delta\right) = 4\pi G a^2 \rho_{dm}\delta\left(1 + \frac{4}{3}\frac{\rho_r}{\rho_{dm}}\right) = 4\pi G a^2 \rho_{dm}\delta\left(1 + \frac{4}{3}\frac{a_{eq}}{a}\right) =$$

$$= 4\pi G a^2 \rho_{dm}\delta\left(1 + \frac{4}{3y}\right)$$

$\Longrightarrow$ Einstein equation becomes $3\frac{\dot a}{a}\left(\dot\Phi + \frac{\dot a}{a}\Phi\right) = 4\pi G a^2 \rho_{dm}\delta\left(1 + \frac{4}{3y}\right)$

$$3\frac{\dot{a}}{a}\left(\dot{\Phi}+\frac{\dot{a}}{a}\Phi\right)=4\pi G a^2 \rho_{dm}\,\delta\left(1+\frac{4}{3y}\right)$$

y new evolution variable, replaces η, a (ignoring baryons!)

So since $\delta = 3\Theta_{r,0}$ we have reduced the problem to equations for δ and Φ → turn 2 1st order equations $(\dot{\delta}, \dot{\Phi})$ into 1 2nd order equation $(\ddot{\Phi})$

Change variable to y

$$\dot{} = \frac{d}{d\eta} = \frac{dy}{d\eta}\frac{d}{dy} = \frac{d}{d\eta}\left(\frac{a}{a_{eq}}\right)\frac{d}{dy} = \frac{1}{a_{eq}}\frac{da}{d\eta}\frac{d}{dy} = \frac{\dot{a}}{a_{eq}}\frac{d}{dy} =$$

$$\dot{a} = \frac{da}{d\eta} = a\frac{da}{dt} = a^2\left(\frac{1}{a}\frac{da}{dt}\right) = a^2 H$$

$$= \frac{a^2 H}{a_{eq}}\frac{d}{dy} = \underbrace{\left(\frac{a}{a_{eq}}\right)}_{y} aH\frac{d}{dy} = aHy\frac{d}{dy} = aHy\,'$$

$$\underline{\left[\,' \equiv \frac{d}{dy}\right]}$$

$$\dot{} \longrightarrow aHy\,'$$

Einstein equation becomes

$$3\frac{\dot{a}}{a}\left(\dot{\Phi}+\frac{\dot{a}}{a}\Phi\right) = 4\pi G a^2 \rho_{dm}\,\delta\left(1+\frac{4}{3y}\right)$$

$$\frac{\dot{a}}{a} = a\left(\frac{1}{a}\frac{da}{dt}\right) = aH$$

$$y = \frac{\rho_{dm}}{\rho_r} \qquad y+1 = \frac{\rho_r+\rho_{dm}}{\rho_r} = \frac{\rho}{\rho_r}$$

$$\Rightarrow \frac{1}{y+1} = \frac{\rho_{dm}}{\rho_r}\frac{\rho_r}{\rho} \rightarrow \rho_{dm} = \frac{y}{y+1}\rho$$

$$3aH\left(aHy\Phi' + aH\Phi\right) = 4\pi G\rho a^2\frac{y}{y+1}\delta\left(1+\frac{4}{3y}\right)$$

$$\frac{8\pi G}{3}\rho = H^2 \longrightarrow 4\pi G\rho = \frac{3}{2}H^2$$

$$\cancel{3a^2H^2}y\Phi' + \cancel{3a^2H^2}\Phi = \cancel{\frac{3}{2}a^2H^2}\frac{y}{y+1}\delta\left(1+\frac{4}{3y}\right)$$

$$\Downarrow$$

$$y\Phi' + \Phi = \frac{y}{2(y+1)}\delta\left(1+\frac{4}{3y}\right) = \delta\frac{y+\frac{4}{3}}{2(y+1)} = \delta\frac{3y+4}{6(y+1)}$$

We want to turn 2 1st order equations into 1 2nd order equation

- $$\begin{cases} \delta' = -3\Phi' \\ y\Phi' + \Phi = \dfrac{3y+4}{6(y+1)}\,\delta \end{cases}$$

$$\delta' = \left[\frac{6(y+1)}{3y+4}\left[y\Phi' + \Phi\right]\right]' = -3\Phi'$$

... a mess!

- $$\frac{d}{dy}\left[\frac{6(y+1)}{3y+4}\left(y\Phi' + \Phi\right)\right] = \frac{6\Phi}{(3y+4)^2} + \frac{48+40y+36y^2}{(3y+4)^2}\Phi' + \frac{y(y+1)}{(3y+4)}\Phi'' = -3\Phi'$$

straightforward but tedious exercise! (I use Mathematica)

(algebra)

$$\frac{6}{(3y+4)^2}\Phi + \frac{3(32+56y+21y^2)}{(3y+4)^2}\Phi' + \frac{6y(y+1)(3y+4)}{(3y+4)^2}\Phi'' = 0$$

- $$\Rightarrow \boxed{\Phi'' + \frac{21y^2+56y+32}{2y(y+1)(3y+4)}\Phi' + \frac{\Phi}{y(y+1)(3y+4)} = 0}$$

change of variable $u \equiv \dfrac{y^3}{\sqrt{1+y}}\Phi$ $\quad u' = \dfrac{y^2\left[(6+5y)\Phi + 2y(1+y)\Phi'\right]}{2(1+y)^{3/2}}$

$$\boxed{u'' + u'\left[-\frac{2}{y} + \frac{3/2}{1+y} - \frac{3}{3y+4}\right] = 0}$$

- Note: no u term! Just u'' and u', so it's a "fake" 2nd order equation, really just 1st order equation for u' Integrable!

$$\frac{d^2u}{dy^2} = \frac{du}{dy}\left[\frac{2}{y} - \frac{3/2}{1+y} + \frac{3}{3y+4}\right]$$

$$\rightarrow \frac{du'}{dy} = u'\left[\frac{2}{y} - \frac{3/2}{1+y} + \frac{3}{3y+4}\right] \Rightarrow$$

$$\rightarrow \int\frac{du'}{u'} = \int dy\left[\frac{2}{y} - \frac{3/2}{1+y} + \frac{3}{3y+4}\right]$$

$$\rightarrow \ln(u') = \text{const} + 2\ln(y) - \frac{3}{2}\ln(1+y) + \ln(3y+4)$$

const

$$\rightarrow \frac{du}{dy} = A\,\frac{y^2(3y+4)}{(1+y)^{3/2}} \qquad \text{Recall} \quad u = \frac{y^3}{\sqrt{1+y}}\,\Phi$$

$$u(y) = \frac{y^3}{\sqrt{1+y}}\,\Phi = A\int_0^y dy'\,\frac{y'^2(3y'+4)}{(1+y')^{3/2}} + \boxed{\text{const}}$$

$u(0)=0$ since $\frac{y^3\Phi}{\sqrt{1+y}} \longrightarrow 0$ as $y\to 0$

[$y\to 0$ means early times]

small y $(y\to 0)$

$$y^3\Phi \simeq A\int_0^y dy'\,4y'^2 = \frac{4A}{3}y^3 \Rightarrow \Phi \simeq \frac{4A}{3}$$

$$\lim_{y\to 0}\Phi = \frac{4A}{3} \Rightarrow A = \lim_{y\to 0}\frac{3}{4}\Phi = \frac{3\Phi(0)}{4}$$

$$\Rightarrow \Phi = \frac{3\Phi(0)}{4}\left(\frac{y^3}{\sqrt{1+y}}\right)^{-1}\int_0^y dy'\,\frac{y'^2(3y'+4)}{(1+y')^{3/2}} = \cancel{\frac{3\Phi(0)}{4}\frac{y^3}{\sqrt{1+y}}}$$

I use Mathematica

$$\frac{2}{15}\left(16 + \frac{-16+y(-8+y(2+9y))}{\sqrt{1+y}}\right)$$

Exercise!

Hint: substitution $x \equiv \sqrt{1+y}$

$$= \frac{\cancel{3}}{\cancel{4}_2}\, \Phi(0)\, \frac{\sqrt{1+y}}{y^3}\, \frac{\cancel{2}}{\cancel{15}_5} \left(16 + \frac{y\left[y(2+9y)-8\right]-16}{\sqrt{1+y}}\right) =$$

$$= \frac{\Phi(0)}{10}\, \frac{1}{y^3} \left(16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\right)$$

~~Final expr~~

$$\boxed{\Phi = \frac{\Phi(0)}{10}\, \frac{1}{y^3} \left(16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\right)}$$

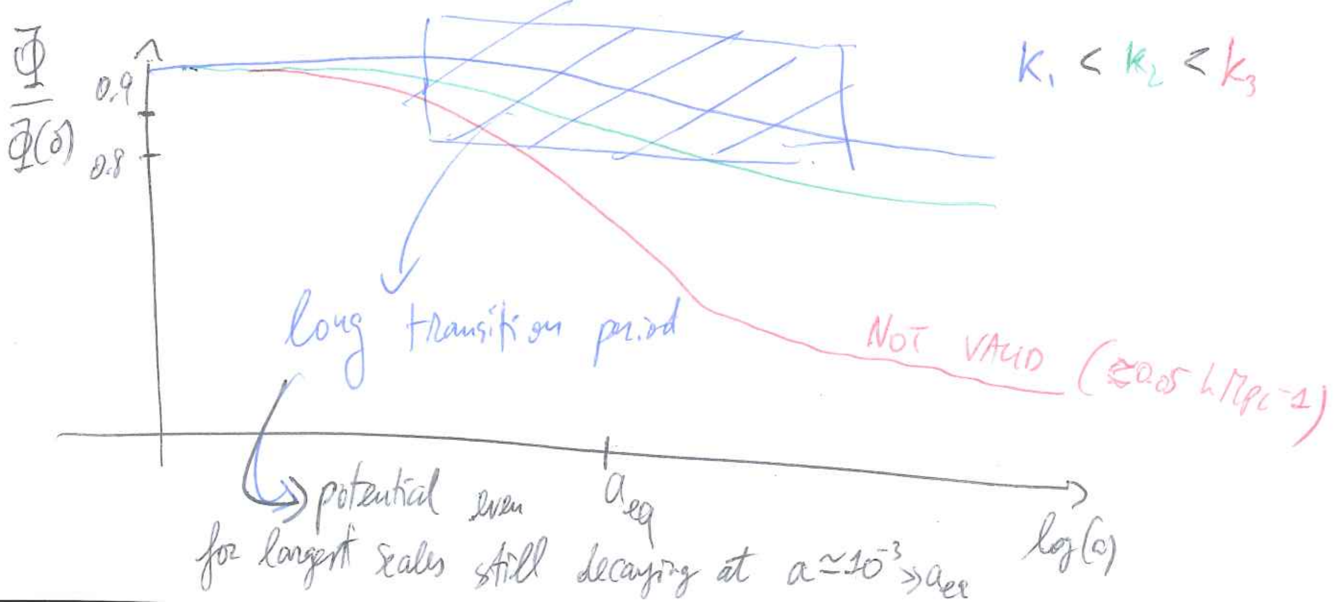
Gravitational potential
- for large-scale modes
- on super-horizon scales (valid even deep MD)
- $y = \frac{a}{a_{eq}} \propto a$
- ignoring baryons

Properties:

- $y \to 0$ ($a \to 0$, early times) $\Longrightarrow \Phi \simeq \Phi(0)$ as expected!
- $y \gg 1$ ($a \gg a_{eq}$, deep MD) $\Longrightarrow \Phi \simeq \frac{\Phi(0)}{10} \frac{1}{y^3} 9y^3 = \frac{9}{10}\Phi(0)$

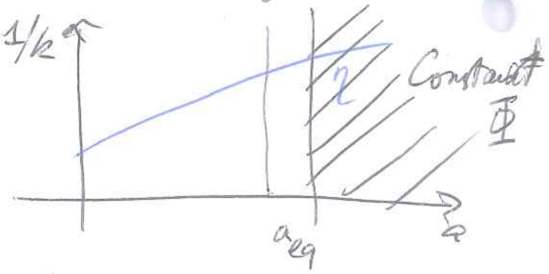
So even if the mode is super-horizon (no causal physics operates) the potential still drops by 10% when going through matter-radiation equality ~~($k \lesssim 0.01\ h\,\mathrm{Mpc}^{-1}$)~~ (the RD → MD transition is long)

Solution works quite well down to $k \lesssim 0.01\ h\,\mathrm{Mpc}^{-1}$,



Horizon crossing

We will prove that when the mode crosses the horizon during MD, Φ remains constant (as long as MD holds)



This time we cannot neglect k terms, but we can neglect radiation. Full set of equations (use no-derivatives Einstein equation.)

$$\cancel{\dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi}}$$ neglect radiation

$$\cancel{\dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi}$$ neglect radiation

$$\dot{\delta} + ikv = -3\dot{\Phi}$$

$$\dot{v} + \frac{\dot{a}}{a}v = ik\Phi$$

$$k^2\Phi = 4\pi G a^2\left[\rho_{dm}\delta + \cancel{4\rho_r\Theta_{r,0}} + \frac{3aH}{k}\left(i\rho_{dm}v + \cancel{4\rho_r\Theta_{r,1}}\right)\right]$$

$\frac{8\pi G\rho}{3} = H^2$

$$4\pi G a^2\rho_{dm}\left[\delta + \frac{3aHiv}{k}\right] \overset{MD}{\simeq} 4\pi G a^2\rho\left[\delta + \frac{3aHiv}{k}\right] = \frac{3}{2}a^2H^2\left[\delta + \frac{3aHiv}{k}\right]$$

$$\Longrightarrow \begin{cases} \dot{\delta} + ikv = -3\dot{\Phi} \\ \dot{v} + \frac{\dot{a}}{a}v = ik\Phi \\ k^2\Phi = \frac{3}{2}a^2H^2\left(\delta + \frac{3aHiv}{k}\right) \end{cases}$$

plug in

$$\delta = \frac{2}{3}\frac{k^2\Phi}{a^2H^2} - \frac{3aHiv}{k}$$

$$\dot{\delta} + ikv = -3\dot{\Phi} \longrightarrow \frac{d}{d\eta}\left[\frac{2}{3}\frac{k^2\Phi}{a^2H^2} - \frac{3aHiv}{k}\right] + ikv = -3\dot{\Phi}$$

Let's see if $\Phi = \text{const}$ is a solution! $\longrightarrow$ prove $\Phi = \text{const}$

$$\frac{d}{d\eta}\left[\frac{2}{3}\frac{k^2\Phi}{a^2H^2} - \frac{3aHiv}{k}\right] + ikv = 0$$

$$\frac{d\left[(aH)^{-2}\right]}{d\eta} = \frac{d}{d\eta}\left(\frac{\eta^2}{4}\right) = \frac{\eta}{2} = \frac{1}{aH}$$

Useful relations during MD

$$aH = \frac{2}{\eta} \longrightarrow \frac{d(aH)}{d\eta} = -\frac{2}{\eta^2} = -\frac{a^2H^2}{2} \qquad \frac{d\left[(aH)^{-1}\right]}{d\eta} = \frac{1}{2} \qquad \frac{d\left[(aH)^2\right]}{d\eta} = 2aH\frac{d(aH)}{d\eta} = -a^3H^3$$

So we just need to expand

$$\frac{d}{d\eta}\left[\frac{2}{3}\frac{k^2\Phi}{a^2H^2} - \frac{3aHiv}{k}\right] + ikv = 0$$

(use the velocity equation to eliminate $\dot{v}$, and check if $\dot{\Phi}=0$ is a solution

$$\frac{2k^2}{3a^2H^2}\dot{\Phi} + \frac{2}{3}k^2\Phi \underbrace{\frac{d\left[(aH)^{-2}\right]}{d\eta}}_{1/aH} - \frac{3aHi\dot{v}}{k} - \frac{3iv}{k}\underbrace{\frac{d(aH)}{d\eta}}_{-a^2H^2/2} + ikv =$$

$$= \frac{2k^2\dot{\Phi}}{3a^2H^2} + \frac{2k^2\Phi}{3aH} - \frac{3aHi\dot{v}}{k} + \frac{3a^2H^2iv}{2k} + ikv =$$

$\dot{v} = ik\Phi - \frac{\dot{a}}{a}v = $
$= ik\Phi - aHv$

$$= \frac{2k^2\dot{\Phi}}{3a^2H^2} + \frac{2k^2\Phi}{3aH} + 3aH\Phi + \frac{3a^2H^2iv}{k} + \frac{3a^2H^2iv}{2k} + ikv =$$

$$= \frac{2k^2\dot{\Phi}}{3a^2H^2} + \frac{2k^2\Phi}{3aH} + 3aH\Phi + \frac{9}{2}\frac{a^2H^2iv}{k} + ikv =$$

$$= \boxed{\frac{2k^2\dot{\Phi}}{3a^2H^2} + \left(\frac{iv}{k} + \frac{2\Phi}{3aH}\right)\left(\frac{9a^2H^2}{2} + k^2\right) = 0}$$

differential equation for Φ large-scale, sub-horizon modes

Take derivatives in η, use velocity equation to eliminate $\dot{v}$ → ideally get equation for $\ddot{\Phi}$

Goal: show that equation is of the form

$$\alpha\ddot{\Phi} + \beta\dot{\Phi} = 0$$

↳ Φ = const solution.

Prove that terms other than $\ddot{\Phi}, \dot{\Phi}$ are 0!

$$\frac{d}{d\eta}\left[\frac{2k^2\dot{\Phi}}{3a^2H^2} + \frac{2k^2\Phi}{3aH} + 3aH\Phi + \frac{9}{2}\frac{a^2H^2 iv}{k} + ikv\right] =$$

look only at no-derivatives in Φ terms

$$= \cancelto{\ddot{\Phi}}{\frac{2k^2\ddot{\Phi}}{3a^2H^2}} + \cancelto{\dot{\Phi}}{\frac{2k^2}{3}\dot{\Phi}\frac{d[(aH)^{-2}]}{d\eta}} + \cancelto{\dot{\Phi}}{\frac{2}{3aH}k^2\dot{\Phi}} + \frac{2k^2\Phi}{3}\underbrace{\frac{d[(aH)^{-1}]}{d\eta}}_{1/2} + \cancelto{\dot{\Phi}}{3aH\dot{\Phi}} + 3\Phi\underbrace{\frac{d(aH)}{d\eta}}_{-a^2H^2/2}$$

$$+ \frac{9}{2}\frac{a^2H^2}{k}i\dot{v} + \frac{9}{2}\frac{iv}{k}\underbrace{\frac{d[(aH)^2]}{d\eta}}_{-a^3H^3} + ik\dot{v} =$$

$\dot{v} = ik\Phi - aHv$

$$= \frac{k^2\Phi}{3} - \frac{3a^2H^2}{2}\Phi + \frac{9}{2}\frac{a^2H^2}{k}i\dot{v} - \frac{9}{2}\frac{a^3H^3 iv}{k} + ik\dot{v} =$$

$$= \frac{k^2\Phi}{3} - \frac{3a^2H^2}{2}\Phi - \frac{9}{2}a^2H^2\Phi - \frac{9}{2}\frac{a^3H^3 iv}{k} - \frac{9}{2}\frac{a^3H^3 iv}{k} - k^2\Phi - ikaHv =$$

$$= -\frac{2}{3}k^2\Phi - ikaHv - 9\frac{a^3H^3 iv}{k} - 6a^2H^2\Phi =$$

$$= -\left(\frac{iaHv}{k} + \frac{2\Phi}{3}\right)\left(9a^2H^2 + k^2\right) \propto \dot{\Phi}$$

since $$\frac{2k^2\dot{\Phi}}{3a^2H^2} + \underbrace{\left(\frac{iv}{k} + \frac{2\Phi}{3aH}\right)}_{\propto\dot{\Phi}}\left(\frac{9a^2H^2}{2} + k^2\right) = 0$$

So Φ equation has only $\ddot{\Phi}$ and $\dot{\Phi}$, no Φ term

$\alpha\ddot{\Phi} + \beta\dot{\Phi} = 0 \longrightarrow \Phi = \text{const}$ is a solution

But initial conditions pick up $\Phi = \text{const} \longrightarrow \Phi = \text{const}$ is the solution

$\Phi \sim \text{const}$ as long as MD holds, then decay due to DE, but this decay is described by $D(a)$

Conclusion: for $k \ll a_{eq} H(a_{eq}) \simeq 0.073\, \Omega_m h^2\, \mathrm{Mpc}^{-1}$

- $T(k) \sim 1$

Transfer function is very close to 1!

$T(k) = T\left(\frac{k}{k_{eq}}\right)$ $\xrightarrow{\text{since } y = \frac{a}{a_{eq}}}$ large-scale $P(k)$ probes initial conditions $\to \Phi_p(k)$

Note: $\Phi \sim \text{const} \longrightarrow \delta \propto a$

Poisson equation

$$k^2 \Phi \sim 4\pi G a^2 \rho_{dm} \delta \propto a^2 a^{-3} \delta = \frac{\delta}{a} = \text{const}$$

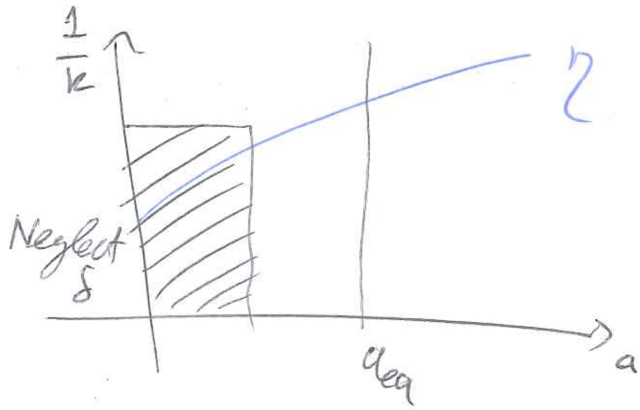
$\rho_{dm} \propto a^{-3}$

$\Longrightarrow \boxed{\delta \propto a}$ during MD era

Small scales

Modes cross deep during RD, then to MD once sub-horizon

Horizon crossing



- Φ mostly determined by r which influences δ but is not influenced by it

$$\Theta_0, \Theta_1 \rightleftarrows \Phi \longrightarrow \delta, v \qquad \text{"}\to\text{"} = \text{influences}$$

Strategy:

- solve for Θ_0, Θ_1, Φ
- solve for δ using Φ as external driving force