Summary

- <u>Large-scale super-horizon solution</u> (drop k terms)

Assumes $k\eta \ll 1$, mode crosses horizon deep in MD, so valid for $a \ll a_{eq}$, $a \sim a_{eq}$, $a \gg a_{eq}$

$\Phi \sim$ constant but drops by $\frac{1}{10}$ $\left(\Phi \to \frac{9}{10}\Phi\right)$ at matter-radiation equality even if mode is super-horizon

- <u>Large-scale horizon crossing solution</u> (Φ constant)

Valid for $a \gg a_{eq}$, $k\eta \ll 1$

$\Phi \sim$ Constant is also a solution
$\delta \propto a$

valid for $a \gg a_{eq}$

- <u>Small-scale horizon crossing solution</u> (neglect δ)

Valid for $a \ll a_{eq}$, $k\eta \gg 1$, mode crosses deep during RD.

$$\Phi = 3\Phi_p \left(\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3} \right)$$ decay + oscillations

$\delta(k,\eta) = A\Phi_p \ln(Bk\eta)$ logarithmic growth

- <u>Small-scale sub-horizon solution</u> (neglect Θ_r)

Valid for $a \sim a_{eq}$, $a \gg a_{eq}$, $k\eta \gg 1$

Meszaros equation

$\delta(y)$

$D_1(y) = y + \frac{2}{3}$ $\quad y = \frac{a}{a_{eq}}$ $\quad D_1(a) \propto a$

$D_2(y) = D_1(y) \ln\left[\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right] - 2\sqrt{1+y}$ $\quad \propto y^{-3/2}$ (late times)

Now we can explore them...

Numerical results and fits

- Splining everything together we can get the transfer function

On small scales $\delta(k,y) = C_1 D_1(y) + C_2 D_2(y)$ can find C_1 explicitly (see Dodelson)

$$\rightarrow \delta(k,a) = \frac{3A\Phi_P(k)}{2} \ln\left[\frac{4Be^{-3}a_{eq}}{a_H}\right] D_1(a)$$

where $a_H H = K$, a_H scale factor when k crosses horizon

$$\frac{a_{eq}}{a_H} = \frac{\sqrt{2}k}{k_{eq}}$$

Recalling definition of transfer function

$$\delta(k,a) = \frac{3}{5}\frac{k^2}{\Omega_m H_0^2}\Phi_P(k)T(k)D_1(a)$$

$$\Rightarrow T_1(k) = \frac{5\,\delta(k,a)}{3k^2}\frac{\Omega_m H_0^2}{\Phi_P(k)D_1(a)} = \frac{5}{2}\frac{A\Omega_m H_0^2}{k^2 a_{eq}} \ln\left[\frac{4Be^{-3}\sqrt{2}k}{k_{eq}}\right]$$

for $k \gg k_{eq}$

$$\boxed{T(k) \simeq 12\frac{k_{eq}^2}{k^2}\ln\left[\frac{k}{8k_{eq}}\right]} \quad \text{for } k \gg k_{eq}$$

So on small scales $T(k) \propto \frac{\ln k}{k^2}$

and $P(k) \propto P_{prim}(k)T^2(k) \propto k^{n_s}T^2(k) \propto \frac{\ln k}{k^{4-n}} \propto \frac{\ln k}{k^3}$ for $n_s \sim 1$

on large scales $P(k) \propto k^{n_s} \sim k$

Growth function

At late times ($z \lesssim 10$) all modes we care about entered horizon.

But we can use ~~δ~~ $y \gg 1$ limit of Meszaros equation only if $\Omega_m = 1$, in which case all modes experience the same growth factor (k does not enter). When DE dominates we need to generalize this

We started from: (for $\Omega_m = 1$)

$$\begin{cases} \delta' + \dfrac{ikv}{aHy} = -3\Phi' \\ v' + \dfrac{v}{y} = \dfrac{ik\Phi}{aHy} \\ k^2\Phi = \dfrac{3y}{2(y+1)} a^2 H^2 \delta \end{cases}$$

$\downarrow$ Exercise $\quad$ ($y \to a$, neglect radiation)

$$\boxed{\frac{d^2\delta}{da^2} + \left(\frac{d\ln(H)}{da} + \frac{3}{a}\right)\frac{d\delta}{da} - \frac{3\Omega_m}{2a^2}\frac{H_0^2}{H^2}\delta = 0}$$

Two solutions:
- $\delta \propto H$ but this is decaying since $\dfrac{dH}{da} < 0$
- growing mode

Try $u \equiv \dfrac{\delta}{H}$

$$\rightarrow \frac{d^2u}{da^2} + 3\left[\frac{d\ln(H)}{da} + \frac{1}{a}\right]\frac{du}{da} = 0$$

Integrate: $v \equiv \dfrac{du}{da}$ $\quad \dfrac{dv}{da} + 3\left(\dfrac{d\ln(H)}{da} + \dfrac{1}{a}\right)v = 0$

$$\Rightarrow \frac{dv}{v} = -3\int \frac{d\ln(H)}{da}\,da - 3\int\frac{da}{a} = -3\ln H - 3\ln a = \ln\left((aH)^{-3}\right)$$

$$\Rightarrow \frac{du}{da} \propto (aH)^{-3} \quad \Rightarrow u(a) \propto \int_0^a \frac{da'}{(a'H(a'))^3} = \frac{\delta}{H}$$

$$\Rightarrow \boxed{\delta \propto D_1(a) \propto H(a)\int_0^a \frac{da'}{[a'H(a')]^3}}$$

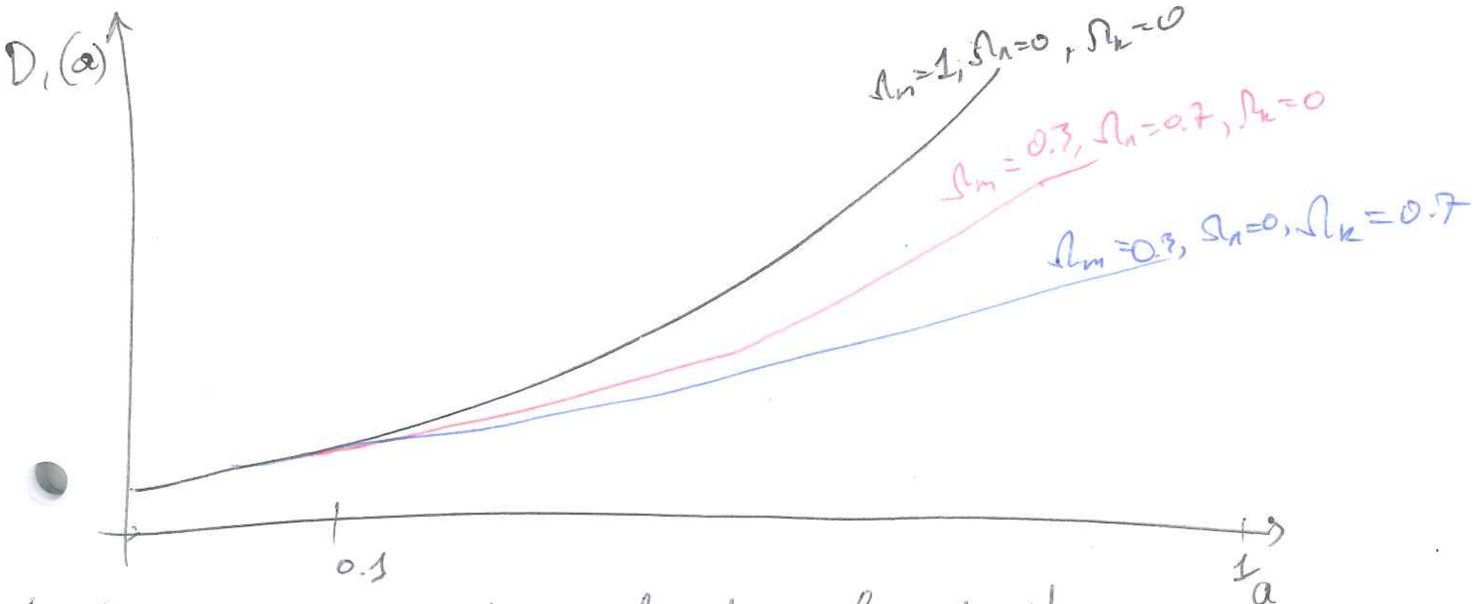
Fixing the proportionality constant

$$D_1(a) = \frac{5\Omega_m}{2}\frac{H(a)}{H_0}\int_0^a \frac{da'}{\left(\frac{a'H(a')}{H_0}\right)^3}$$

Check flat, MD Universe

$$\delta \propto H(a)\int_0^a \frac{da'}{[a'H(a')]^3} \propto a^{-3/2}\int\frac{da'}{[a'a'^{-3/2}]^3} \propto a^{-3/2}\int\frac{da'}{a'^{-3/2}}$$

$$\propto a^{-3/2}a^{5/2} \propto a \qquad \delta \propto a \quad \text{as we knew}$$



Dark energy slows down structure formation!

Beyond cold DM

Here we approximated $m \simeq DM$. But there is also:

- baryons
  - → suppress $P(k)$ on small scales (pressure) → b-γ coupling
  - → acoustic oscillations
- massive neutrinos
  - → power suppression on small scales
- Dark energy
  - → turnover in $P(k)$ indirect effect due to lower Ω_m
  - → $\delta \propto \dfrac{\Phi}{\Omega_m}$ raises $P(k)$ at Φ fixed
  - → Growth of $\delta \propto D(a)$ affected

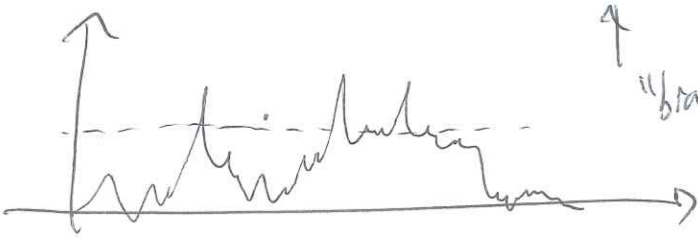
Note: we don't measure $P_m(k)$, but $P_t(k)$

"tracer"

→ galaxies
QSOs
Ly-α ...

On large scales $P_t(k) \simeq b\, P_m(k)$

"bias"



ANISOTROPIES

- Previously studied matter inhomogeneities (δ), now care about photon anisotropies (CMB)

Overview

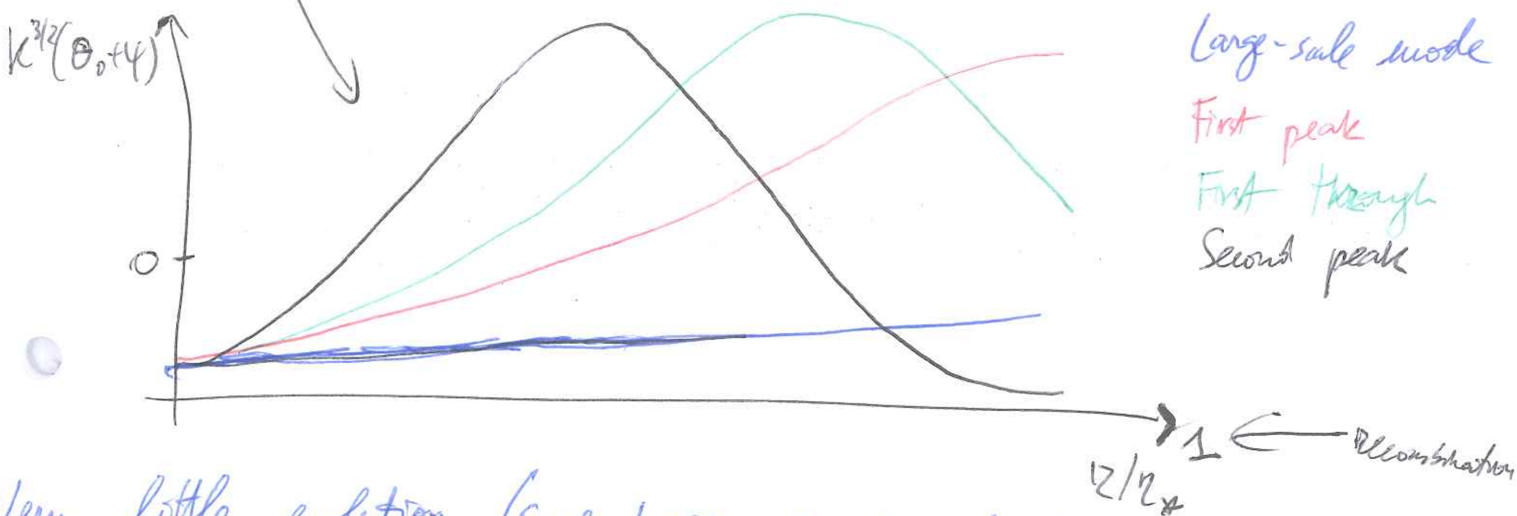
A posteriori, look at $k^{3/2}(\underbrace{\Theta_0 + \Psi})$

Amplitude of perturbations $\propto k^{-3/2}$

Ψ accounts for redshift when climbing out of overdensities ($\Psi < 0$) or blueshift when entering underdensities ($\Psi > 0$)

So we really observe $\Theta_0 + \Psi$

- General feature: photon perturbations grow very little, remain quite linear (unlike δ)



Very little evolution (super-horizon, no causal physics)

Reaches maximum at recombination → expect large fluctuations on corresponding scales (for anisotropies)

Amplitude at recombination zero (has done exactly half oscillation) expect small fluctuations

- One full oscillation by recombination

→ signatures of acoustic oscillations

So we expect a series of peaks and troughs in CMB anisotropy spectrum for smaller and smaller scales whose modes entered horizon earlier

Example: $k^{3/2}|\Theta_0+\Psi|(\eta_*)$ vs $k\eta_0$ (snapshot at recombination)

If raise Ω_b (amount of b) odd peaks are higher than even peaks, and small-scale perturbations ($k\eta_0 \gtrsim 500$) are damped

Cartoon version of equations

$$\ddot{\Theta}_0 + k^2 c_s^2 \Theta_0 = F$$

pressure (restoring force)

~~restoring~~ driving force

Forced harmonic oscillator

Review of forced harmonic oscillator

$$F = F_0 - kx \longrightarrow \ddot{x} + \frac{k}{m}x = +\frac{F_0}{m}$$

drives to large x

restores to small x ($x \to 0$)

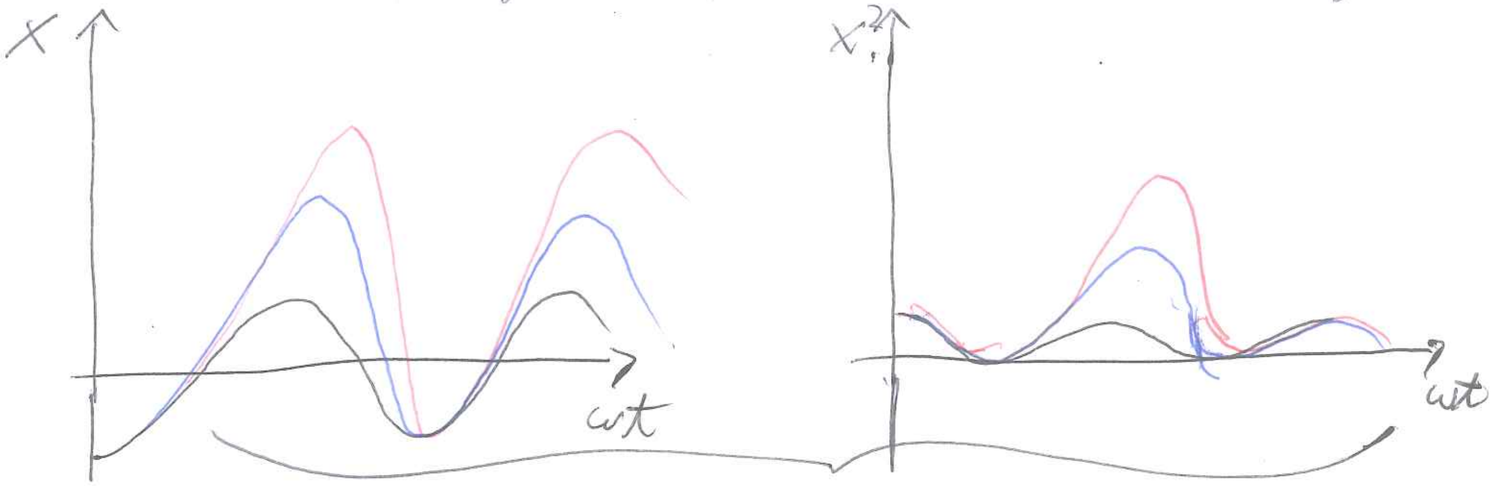
Full solution (if oscillator initially at rest)

$$x = A\cos(\omega t) + \frac{F_0}{m\omega^2} \qquad \omega^2 = \frac{k}{m}$$

$$= A\cos(\omega t) + \frac{F_0}{k}$$

Effectively driving force sets oscillations around a new zero-point $x \geq 0$ (if $F_0 > 0$)

- For lower ω (higher m / lower k) zero-point shift larger



- Unforced
  ↳ oscillations symmetric around origin
  even/odd peaks identical in x^2!

Forced (high ω) Forced (low ω)
↳ odd peaks higher than even peaks

Peaks at $t = \frac{n\pi}{\omega}$, identical height in unforced case, odd peaks higher in forced case, especially for lower frequencies

- Even peaks correspond to negative positions (against driving force)

So $\ddot{\Theta}_0 + k^2 c_s^2 \Theta_0 = F$ explains

oscillations

larger even/odd disparity as Ω_b is raised, since this lowers c_s^2 (~~raises~~ lowers ω)

Peaks at $\frac{n\pi}{\omega}$ shifted to larger k as $\Omega_b \uparrow$ $c_s^2 \downarrow$ and

- spacing between peaks gets larger

(think: $\frac{n\pi}{\omega} \propto \frac{1}{\omega} \propto \frac{1}{k^2 c_s^2}$ $\longrightarrow$ $c_s^2 \downarrow$ $k \uparrow$)

Another way to understand this:

- first peak has been growing since entered horizon

$\Omega_b \uparrow \Rightarrow c_s^2 \downarrow \Rightarrow P \downarrow \Rightarrow$ growth easier (peak is higher!)
more overdense

- second peak corresponds to underdensity

$\Omega_b \uparrow \Rightarrow c_s^2 \downarrow \Rightarrow P \downarrow \Rightarrow$ harder to escape well (peak is lower)
less underdense

Increasing Ω_b also damps the oscillations!

Photons have finite mean free path

$$\lambda_{MFP} \propto (n_e \sigma_T)^{-1}$$

scatters in Hubble time $\quad \frac{n_e \sigma_T}{H} = \frac{\lambda_H}{\lambda_{MFP}}$

Cosmological photon over Hubble time moves

$$\lambda_D \sim \lambda_{MFP} \sqrt{N} = \lambda_{MFP} \sqrt{n_e \sigma_T H^{-1}} = \frac{1}{n_e \sigma_T} \sqrt{\frac{n_e \sigma_T}{H}} = \frac{1}{\sqrt{n_e \sigma_T H}}$$

Perturbations on scales $\lesssim \lambda_D$ are washed out (high k modes damped)

$$\lambda_D \propto \frac{1}{\sqrt{n_e}} \propto \frac{1}{\sqrt{\Omega_b}} \Rightarrow k_D \propto \sqrt{\Omega_b}$$

when Universe ionized

So models with more baryons $\Omega_b \uparrow$ have $k_D \uparrow$ (damping occurs on smaller scales)

$k^{3/2} |\Theta_0 + \Psi| (\eta_*)$ gives perturbations at recombination but we observe them today!

Photons from hot / cold spots typically separated by comoving distance k^{-1} $\rightarrow$ angular separation $\theta \simeq \frac{k^{-1}}{\eta_0 - \eta_*} = \frac{1}{k(\eta_0 - \eta_*)}$

In multiple moments

$\theta \sim \frac{1}{\ell}$

note: $\eta_0 \simeq 14$ Gpc *

$\eta_* \ll \eta_0$

$$\rightarrow \frac{1}{\ell} \sim \frac{1}{k(\eta_0 - \eta_*)} \sim \frac{1}{k\eta_0} \Rightarrow \ell \simeq k\eta_0$$

$\cancel{\ell \simeq 1.4\times10^4 \left(\frac{k}{\mathrm{Mpc}}\right)}$

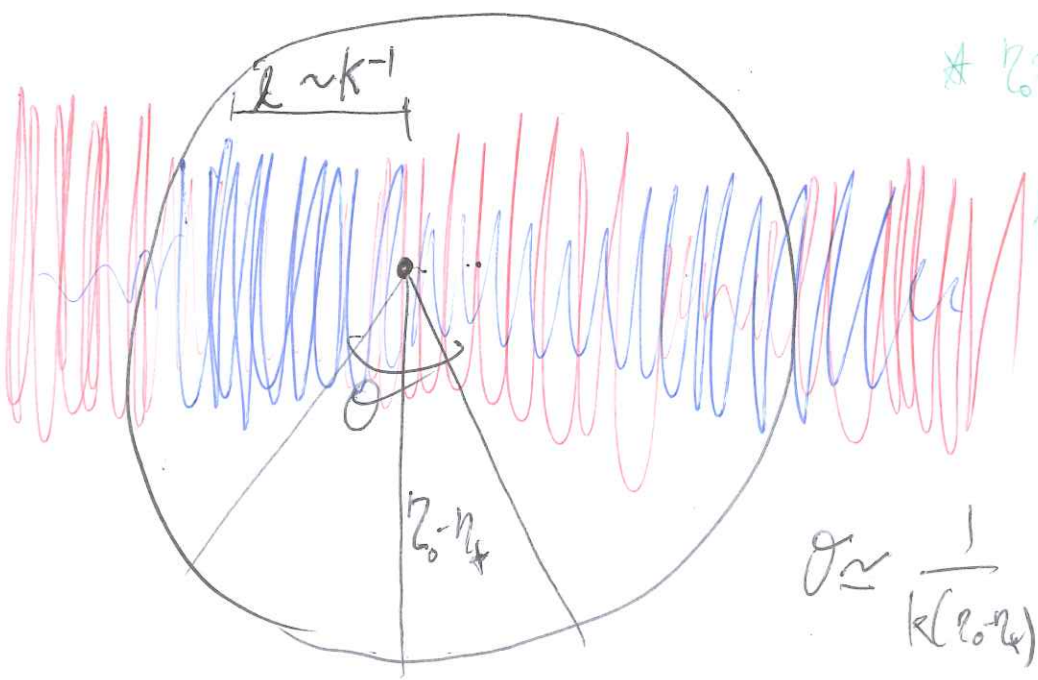
$\cancel{\ell \simeq 1.4\times10^4 \frac{k}{\mathrm{Mpc}}}$

So inhomogeneities on scales k are projected roughly into anisotropies on angular scales $k\eta_0$

Modulo a few things which happen to photons on their journey from last-scattering to us

$\left\{\begin{array}{l} \phi \neq \text{const} \\ \text{Universe not flat?} \\ \text{lensing} \end{array}\right]$ Secondary anisotropies (~10% effect to anisotropies)

Pictorially, perturbation with wavenumber k



$\theta \simeq \frac{1}{k(\eta_0 - \eta_*)}$

* $\eta_0 \simeq 14$ Gpc $= 1.4\times10^4$ Mpc

$\ell \simeq 1.4\times10^4 \left(\frac{k}{\mathrm{Mpc}^{-1}}\right)$

Large-scale anisotropies

Recall equation for photon perturbations

$$\begin{cases} \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \\ \dot{\Theta}_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\Phi \end{cases}$$

With initial conditions $\Phi(k,\eta_i) = -\Psi(k,\eta_i) = 2\Theta_0(k,\eta_i)$

$$\Theta_1(k,\eta_i) = -\frac{k\,\Phi(k,\eta_i)}{6aH}$$

Let's focus on the large-scale, super-horizon limit of the first equation

$k\eta \ll 1$ $\left(\dot{\Theta}_0 \sim \frac{\Theta_0}{\eta}\right)$ so $\frac{\dot{\Theta}_0}{k\Theta_1} \sim \frac{1}{k\eta} \gg 1$)

$$\dot{\Theta}_0 + \cancel{k\Theta_1} = -\dot{\Phi} \Rightarrow \dot{\Theta}_0 = -\dot{\Phi} \Rightarrow \Theta_0 = -\Phi + \text{const}$$

$\Theta_0 = -\Phi + \text{const}$ but we know $\Theta_0(k,\eta_i) = \frac{\Phi(k,\eta_i)}{2}$

$$\Rightarrow \text{const} = \frac{3\Phi_p}{2} \quad \text{so} \quad \Theta_0(k,\eta_i) = -\Phi_p + \frac{3\Phi_p}{2} = \frac{\Phi_p}{2} = \frac{\Phi(k,\eta_i)}{2}$$

$$\Theta_0 = -\Phi + \frac{3\Phi_p}{2}$$

We already got an expression for the large-scale evolution of Φ earlier (for modes that cross the horizon during MD and therefore are still super-horizon during RD)

$$\Phi = \frac{\Phi_p}{10}\frac{1}{y^3}\left[16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\right] \qquad y \equiv \frac{a}{a_{eq}}$$

$y \ll 1$: Φ_p

$y \gg 1$: $\frac{9\Phi_p}{10}$

Recombination takes place sufficiently after equality that we can use the $y \gg 1$ limit $\Phi \to \frac{9\Phi_p}{10}$ (at $\eta = \eta_*$)

Therefore $\Phi(k, \eta_*) = \frac{9}{10}\Phi_p(k) \to \Phi_p(k) = \frac{10}{9}\Phi(k, \eta_*)$

$$\Theta_0(k, \eta_*) = -\Phi(k, \eta_*) + \frac{3\Phi_p(k)}{2} = -\Phi(k, \eta_*) + \frac{\cancel{3}}{\cancel{2}}\frac{\cancel{10}^{5}}{\cancel{9}_{3}}\Phi(k, \eta_*) =$$

$$= -\Phi(k, \eta_*) + \frac{5}{3}\Phi(k, \eta_*) = \frac{2}{3}\Phi(k, \eta_*)$$

$$\Rightarrow \Theta_0(k, \eta_*) = \frac{2}{3}\Phi(k, \eta_*)$$

We argued that the observed anisotropy is $\sim \Theta_0 + \Psi$ to account for photon relative red/blueshift if $\Psi < 0 / \Psi > 0$ since they have to travel out of under/over-densities at recombination

$$\Theta_0 + \Psi \sim \Theta_0 - \Phi \quad (\text{as } \Psi \sim -\Phi)$$

$$(\Theta_0 + \Psi)(k, \eta_*) \approx (\Theta_0 - \Phi)(k, \eta_*) = \frac{2}{3}\Phi(k, \eta_*) - \Phi(k, \eta_*) =$$

$$= -\frac{1}{3}\Phi(k, \eta_*) \approx \frac{1}{3}\Psi(k, \eta_*)$$

So ~~the~~ observed anisotropy $\sim \frac{1}{3}\Psi(k, \eta_*)$

Can also think in terms of density field. Recall

$\dot{\delta} + ikv = -3\dot{\Phi}$ with initial conditions $\delta(k, \eta_i) = \frac{3}{2}\Phi_p$

large-scale limit

$$\dot{\delta} = -3\dot{\Phi} \Rightarrow \cancel{\delta = -3\Phi + \text{const}}$$

$$\delta(k, \eta_*) = -3\Phi(k, \eta_*) + \text{const}$$

$$\cancel{\delta(k,\eta_*) = -3\Phi(}$$

$$\cancel{\delta(\eta_*) = -3\Phi(\eta_*) + \text{const} \xrightarrow{\eta_*}}$$

$$\delta(\eta) = -3\Phi(\eta) + \text{const} \xrightarrow{\eta \to 0} \delta = -3\Phi_p + \text{const} = \frac{3\Phi_p}{2}$$

$$\rightarrow \text{const} = \frac{3\Phi_p}{2} + 3\Phi_p = \frac{9\Phi_p}{2}$$

$$\delta(\eta_*) = -3\Phi(\eta_*) + \underbrace{\frac{9\Phi_p}{2}}_{\frac{9}{2}\Phi_p = \frac{3}{2}\Phi_p - (-3\Phi_p)} = \frac{3}{2}\Phi_p - 3\left[\Phi(\eta_*) - \Phi_p\right]$$

$$\cancel{\delta(\eta_*) = \frac{3}{2}\Phi_p - 3\left[\Phi(\eta_*) - \Phi_p\right] =}$$

$$\Phi_p = \frac{10}{9}\Phi(\eta_*) \qquad \cancel{\Phi(\eta_*) = \frac{9}{10}\Phi_p}$$

$$\delta(\eta_*) = -3\Phi(\eta_*) + \frac{9\Phi_p}{2} = -3\Phi(\eta_*) + \frac{\cancel{9}}{\cancel{2}}\frac{\cancel{10}^{\,5}}{\cancel{9}}\Phi(\eta_*) =$$

$$\Phi(\eta_*) = \frac{9}{10}\Phi_p \Rightarrow \Phi_p = \frac{10}{9}\Phi(\eta_*)$$

$$= -3\Phi(\eta_*) + 5\Phi(\eta_*) = 2\Phi(\eta_*)$$

$$\Rightarrow \delta(\eta_*) = 2\Phi(\eta_*) \qquad \Phi(\eta_*) = \frac{1}{2}\delta(\eta_*)$$

~~So primor~~ So observed anisotropy is

$$(\Theta_0 + \Psi)(k,\eta_*) = -\frac{1}{3}\Phi(k,\eta_*) = -\frac{1}{3}\frac{1}{2}\delta(\eta_*) = -\frac{1}{6}\delta(\eta_*)$$

Summarizing:

$$\Theta_0(k,\eta_*) = \frac{2\Phi(k,\eta_*)}{3} \xrightarrow{\text{observed anisotropy}} \boxed{(\Theta_0 + \Psi)(k,\eta_*) = \begin{cases} -\frac{1}{3}\Phi(k,\eta_*) \\ \frac{1}{3}\Psi(k,\eta_*) \\ -\frac{1}{6}\delta(k,\eta_*) \end{cases}}$$

Consider

$$(\Theta_0+\Psi)(k,\eta_*) = -\frac{1}{6}\delta(\eta_*)$$

$$\Big\downarrow \text{ Fourier transform...}$$

$$\tilde{\Theta}_0+\tilde{\Psi} < 0 \quad \text{for} \quad \delta>0 \ (\Psi<0)$$

$$\downarrow$$

observed anisotropy of an overdense region is negative! (cold spot)

So for large-scale perturbations cold spots correspond to overdense regions at recombination. These were hotter at recombination ($\Theta_0 > 0$ when $\Psi < 0$), however to travel to us they need to climb out of potential wells, and they lose ~~more~~ energy in a way which more than compensates for their originally being hotter, i.e. $\Theta_0+\Psi<0$ when $\Psi<0$

cold spots → overdensities at η_* hot spots → underdensities at η_*

Factor $\frac{1}{6}$ is interesting

$$\cancel{(\Theta_0+\Psi)}$$

$$\frac{\delta T}{T} \sim |\Theta_0+\Psi| \sim \frac{1}{6}\delta \sim \frac{1}{6}\frac{\delta\rho}{\rho}$$

So $\sim 10^{-5}$ anisotropies correspond to $\sim 6\times10^{-5}$ overdensities, so we can ask the question of whether the observed anisotropies are consistent with the overdensities needed to form structure (yes! In ΛCDM with inflation)