Full set of equations (use algebraic Einstein equations)

$$\begin{cases} \boxed{\begin{aligned} &\dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi} \\ &\dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi \end{aligned}} \\ \dot{\delta} + ikv = -3\dot{\Phi} \\ \dot{v} + \frac{\dot{a}}{a}v = ik\Phi \\ \boxed{k^2\Phi = 4\pi G a^2\left[\rho_{dm}\delta + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}\left(i\rho_{dm}v + 4\rho_r\Theta_{r,1}\right)\right]} \end{cases}$$

start considering only Θ_0, Θ_1, Φ

radiation and potential, drop all matter

$$\begin{cases} \dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi} \\ \dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi \\ k^2\Phi = 4\pi G a^2\left[\cancel{\rho_{dm}\delta} + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}\left(\cancel{i\rho_{dm}v} + 4\rho_r\Theta_{r,1}\right)\right] \end{cases}$$

$$\Rightarrow k^2\Phi = 4\pi G a^2\left(4\rho_r\Theta_{r,0} + \frac{12aH}{k}\rho_r\Theta_{r,1}\right) =$$

$$= 16\pi G\rho_r a^2\left(\Theta_{r,0} + \frac{3aH}{k}\Theta_{r,1}\right) = 6a^2H^2\left(\Theta_{r,0} + \frac{3aH}{k}\Theta_{r,1}\right)$$

$H^2 \simeq \frac{8\pi G}{3}\rho_r \Rightarrow 16\pi G\rho_r \simeq 6H^2$

$$\Rightarrow \Phi \simeq \frac{6a^2H^2}{k^2}\left(\Theta_{r,0} + \frac{3aH}{k}\Theta_{r,1}\right)$$

Useful relations during RD

$$aH = \frac{1}{\eta} \longrightarrow \frac{d(aH)}{d\eta} = -\frac{1}{\eta^2} = -a^2H^2 \qquad \frac{d\left[(aH)^{-1}\right]}{d\eta} = 1$$

$$\frac{d\left[(aH)^2\right]}{d\eta} = 2aH\frac{d(aH)}{d\eta} = -2a^3H^3 \qquad \frac{d\left[(aH)^{-2}\right]}{d\eta} = \frac{d}{d\eta}\eta^2 = 2\eta = \frac{2}{aH}$$

$$\Phi = \frac{6a^2H^2}{k^2}\left(\Theta_{r0} + \frac{3aH}{k}\Theta_{r,1}\right)$$ use it to eliminate Θ_{r10}

in $\begin{cases} a) & \dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi} \\ b) & \dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi \end{cases}$

Let me drop "r" for the moment

$$\Rightarrow \Theta_0 = \frac{k^2\Phi}{6a^2H^2} - \frac{3aH}{k}\Theta_1 = \frac{k^2\eta^2}{6}\Phi - \frac{3}{k\eta}\Theta_1$$

$$\dot{\Theta}_0 = \frac{k^2\eta}{3}\Phi + \frac{k^2\eta^2}{6}\dot{\Phi} + \frac{3}{k\eta^2}\Theta_1 - \frac{3}{k\eta}\dot{\Theta}_1$$

a) $\Rightarrow \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi}$ becomes

$$\frac{k^2\eta}{3}\Phi + \frac{k^2\eta^2}{6}\dot{\Phi} + \frac{3}{k\eta^2}\Theta_1 - \frac{3}{k\eta}\dot{\Theta}_1 + k\Theta_1 = -\dot{\Phi}$$

$$\Rightarrow \boxed{-\frac{3}{k\eta}\dot{\Theta}_1 + k\Theta_1\left(1 + \frac{3}{k^2\eta^2}\right) = -\dot{\Phi}\left(1 + \frac{k^2\eta^2}{6}\right) - \Phi\frac{k^2\eta}{3}}$$

b) $\Rightarrow \dot{\Theta}_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\Phi$ becomes

$$\dot{\Theta}_1 - \frac{k}{3}\left(\frac{k^2\eta^2}{6}\Phi - \frac{3}{k\eta}\Theta_1\right) = -\frac{k}{3}\Phi \Rightarrow \boxed{\dot{\Theta}_1 + \frac{1}{\eta}\Theta_1 = -\frac{k}{3}\Phi\left(1 - \frac{k^2\eta^2}{6}\right)}$$

Now we have two 1st order equations for Φ, Θ_1

$$\begin{cases} -\frac{3}{k\eta}\dot{\Theta}_1 + k\Theta_1\left(1 + \frac{3}{k^2\eta^2}\right) = -\dot{\Phi}\left(1 + \frac{k^2\eta^2}{6}\right) - \Phi\frac{k^2\eta}{3} \\ \dot{\Theta}_1 + \frac{1}{\eta}\Theta_1 = -\frac{k}{3}\Phi\left(1 - \frac{k^2\eta^2}{6}\right) \end{cases}$$

Turn it into 2nd order equation for Φ

by repeatedly differentiating

$$-\frac{3}{k\eta}\dot\theta_1 + k\theta_1\left(1+\frac{3}{k^2\eta^2}\right) = -\dot\Phi\left(1+\frac{k^2\eta^2}{6}\right) - \Phi\frac{k^2\eta}{3}$$

$$\Downarrow \quad \dot\theta_1 + \frac{\theta_1}{\eta} = -\frac{k}{3}\Phi\left(1-\frac{k^2\eta^2}{6}\right) \rightarrow \dot\theta_1 = -\frac{k}{3}\Phi\left(1-\frac{k^2\eta^2}{6}\right) - \frac{\theta_1}{\eta}$$

$$-\frac{3}{k\eta}\left[-\frac{\theta_1}{\eta} - \frac{k}{3}\Phi\left(1-\frac{k^2\eta^2}{6}\right)\right] + k\theta_1\left(1+\frac{3}{k^2\eta^2}\right) + \dot\Phi\left(1+\frac{k^2\eta^2}{6}\right) + \frac{k^2\eta}{3}\Phi = 0$$

$$= \frac{3}{k\eta^2}\theta_1 + \frac{\Phi}{\eta}\left(1-\frac{k^2\eta^2}{6}\right) + k\theta_1 + \frac{3}{k\eta^2}\theta + \dot\Phi\left(1+\frac{k^2\eta^2}{6}\right) + \frac{k^2\eta}{3}\Phi =$$

$$= \frac{3}{k\eta^2}\theta_1 + \frac{\Phi}{\eta} - \frac{k^2\eta}{6}\Phi + k\theta_1 + \frac{3}{k\eta^2}\theta_1 + \dot\Phi + \frac{k^2\eta^2}{6}\dot\Phi + \frac{k^2\eta}{3}\Phi =$$

$$= \dot\Phi + \frac{k^2\eta^2}{6}\dot\Phi + \frac{\Phi}{\eta} + \frac{k^2\eta}{6}\Phi + k\theta_1 + \frac{6}{k\eta^2}\theta =$$

$$= \dot\Phi\left(1+\frac{k^2\eta^2}{6}\right) + \frac{\Phi}{\eta}\left(1+\frac{k^2\eta^2}{6}\right) + k\theta_1\left(1+\frac{6}{k^2\eta^2}\right) = 0$$

$$\Rightarrow \dot\Phi + \frac{\Phi}{\eta} = -\frac{\left(1+\frac{6}{k^2\eta^2}\right)}{\left(1+\frac{k^2\eta^2}{6}\right)}k\theta_1 = -\frac{\cancel{k^2\eta^2+6}}{k^2\eta^2}\frac{6}{\cancel{6+k^2\eta^2}}\cancel{k}\theta_1 = -\frac{6}{k\eta^2}\theta_1$$

$$\Rightarrow \boxed{\dot\Phi + \frac{\Phi}{\eta} = -\frac{6}{k\eta^2}\theta_1} \longrightarrow \begin{cases} \theta_1 = -\frac{k\eta^2}{6}\left(\dot\Phi + \frac{\Phi}{\eta}\right) \\ \dot\theta_1 = -\frac{k}{3}\Phi\left(1-\frac{k^2\eta^2}{6}\right) - \frac{\theta_1}{\eta} = \\ \quad = -\frac{k}{3}\Phi\left(1-\frac{k^2\eta^2}{6}\right) + \frac{k\eta}{6}\left(\dot\Phi + \frac{\Phi}{\eta}\right) \end{cases}$$

Now differentiate and replace θ_1, $\dot\theta_1$

$$\dot\Phi + \frac{\Phi}{\eta} = -\frac{6}{k\eta^2}\theta_1 \Rightarrow \dot\Phi + \frac{\Phi}{\eta} + \frac{6}{k\eta^2}\theta_1 = 0$$

~~Derivative~~ $\cancel{\frac{d}{d\eta}\dot\Phi}$ Now take $\frac{d}{d\eta}$!

$$\frac{d}{d\eta}\left(\dot{\Phi}+\frac{\Phi}{\eta}+\frac{6}{k\eta^2}\Theta_1\right)=\ddot{\Phi}-\frac{\Phi}{\eta^2}+\frac{\dot{\Phi}}{\eta}-\frac{12}{k\eta^3}\Theta_1+\frac{6}{k\eta^2}\dot{\Theta}_1=$$

$$=\ddot{\Phi}-\frac{\Phi}{\eta^2}+\frac{\dot{\Phi}}{\eta}-\frac{12}{k\eta^3}\left[-\frac{k\eta^2}{6}\left(\dot{\Phi}+\frac{\Phi}{\eta}\right)\right]+\frac{6}{k\eta^2}\left[-\frac{k}{3}\Phi\left(1-\frac{k\eta^2}{6}\right)+\frac{k\eta}{6}\left(\dot{\Phi}+\frac{\Phi}{\eta}\right)\right]=$$

$$=\ddot{\Phi}-\cancel{\frac{\Phi}{\eta^2}}+\frac{\dot{\Phi}}{\eta}+\frac{2}{\eta}\dot{\Phi}+\cancel{\frac{2}{\eta^2}\Phi}-\cancel{\frac{2\Phi}{\eta^2}}+\frac{k^2\Phi}{3}+\frac{\dot{\Phi}}{\eta}+\cancel{\frac{\Phi}{\eta^2}}=$$

$$=\ddot{\Phi}+\frac{4}{\eta}\dot{\Phi}+\frac{k^2\Phi}{3}$$

$$\Rightarrow \boxed{\ddot{\Phi}+\frac{4}{\eta}\dot{\Phi}+\frac{k^2\Phi}{3}=0}$$

Remember at very (very!) early times

$\ddot{\Phi}\eta+4\dot{\Phi}=0 \longrightarrow \Phi=\eta^p \quad p=0,-3 \leftarrow$ decaying mode we don't care about it

$\Phi=$const mode excited!

$\rightarrow$ Solve this with constraint / initial condition $\Phi=$const

Analytical selection by defining $u\equiv\Phi\eta \longrightarrow \Phi=\frac{u}{\eta}$, $\dot{\Phi}=\frac{\dot{u}}{\eta}-\frac{u}{\eta^2}$

$$\ddot{\Phi}=\frac{\ddot{u}}{\eta}-\frac{2\dot{u}}{\eta^2}+\frac{2u}{\eta^3}$$

$$\ddot{\Phi}+\frac{4}{\eta}\dot{\Phi}+\frac{k^2}{3}\Phi=\frac{\ddot{u}}{\eta}-\frac{2\dot{u}}{\eta^2}+\frac{2u}{\eta^3}+\frac{4}{\eta^2}\dot{u}-\frac{4}{\eta^3}u+\frac{k^2}{3\eta}u=0$$

$$\Rightarrow \ddot{u}+\frac{2}{\eta}\dot{u}+\left(\frac{k^2}{3}-\frac{2}{\eta^2}\right)u=0$$

It's of the Bessel function form! Recall spherical Bessel functions

lth spherical Bessel function solves

$$\frac{d^2 J_l}{dx^2}+\frac{2}{x}\frac{dJ_l}{dx}+\left[1-\frac{l(l+1)}{x^2}\right]J_l=0$$

$l=0$: $\frac{d^2 J_0}{dx^2}+\frac{2}{x}\frac{dJ_0}{dx}+J_0=0 \qquad J_0(x)=\frac{\sin(x)}{x}$

$l=1$: $\frac{d^2 J_1}{dx^2}+\frac{2}{x}\frac{dJ_1}{dx}+\left(1-\frac{2}{x^2}\right)J_1=0 \qquad J_1(x)=\frac{\sin x - x\cos x}{x^2}$

Solutions: $j_1\left(\frac{k\eta}{\sqrt{3}}\right)$ and $\cancel{n_1\left(\frac{k\eta}{\sqrt{3}}\right)}$ Blows up as $\eta \to 0$

j_1: ~~Sp~~ spherical Bessel function

n_1: spherical Neumann function

$u = \Phi\eta \sim \eta$ at early times

$$\frac{\sin x - x\cos x}{x^2} \sim \frac{x}{3} + O(x^3)$$

~~Sol~~ $u = C j_1\left(\frac{k\eta}{\sqrt{3}}\right) = C\left[\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^2}\right]$

$$\Phi = \frac{u}{\eta} = C\left[\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}\right] \xrightarrow[\eta \to 0]{} \frac{C}{3} = \Phi_p$$

$$\Rightarrow C = 3\Phi_p$$

So full solution is

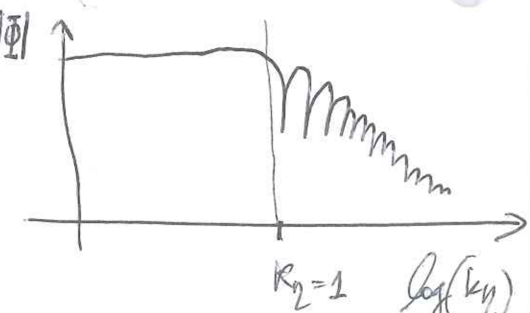
$$\boxed{\Phi = 3\Phi_p\left[\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}\right]}$$

$k\eta \ll 1$ super-horizon $\Phi \propto$ const

$k\eta \gg 1$ sub-horizon

$k\eta \approx 1$ horizon crossing

$\Big\}$ $\Phi \propto$ oscillations $\times$ decay



So as soon as mode crosses horizon during RD the potential starts to decay, then oscillates

decay: dilution of energy density due to expansion

oscillates: due to radiation pressure

Heuristically $\quad \Phi = \frac{6a^2H^2}{k^2}\left(\Theta_{r,0} + \frac{3aH}{k}\Theta_{r,1}\right) \xrightarrow[\substack{k \gg aH \\ k\eta \gg 1}]{\text{sub-horizon}} \Phi \sim \frac{\Theta_0}{\eta^2}$

this is the large $k\eta$ limit of the full solution

So far we have determined Φ, now determine δ

$$\begin{cases} \dot\delta + ikv = -3\dot\Phi \\ \dot v + \frac{\dot a}{a} v = ik\Phi \end{cases}$$

$\longrightarrow$ Differentiate, then eliminate $\dot v$, then eliminate v

$$\frac{d}{d\eta}\left(\dot\delta + ikv\right) = \frac{d}{d\eta}\left(-3\dot\Phi\right) \Rightarrow \ddot\delta + ik\dot v = -3\ddot\Phi$$

$\dot v = -\frac{\dot a}{a}v + ik\Phi$

$$= \ddot\delta + ik\left(-\frac{\dot a}{a}v + ik\Phi\right) = \ddot\delta - ik\frac{\dot a}{a}v - k^2\Phi =$$

$v = -\frac{3\dot\Phi}{ik} - \frac{\dot\delta}{ik} = \frac{3i\dot\Phi}{k} + \frac{i\dot\delta}{k}$

$$= \ddot\delta - i\cancel{k}\frac{\dot a}{a}\left(\frac{3i\dot\Phi}{\cancel{k}} + \frac{i\dot\delta}{\cancel{k}}\right) - k^2\Phi = \ddot\delta + 3\frac{\dot a}{a}\dot\Phi + \frac{\dot a}{a}\dot\delta - k^2\Phi$$

During RD $\quad \frac{\dot a}{a} = \frac{1}{a}\frac{da}{d\eta} = a\left(\frac{1}{a}\frac{da}{dt}\right) = aH = \frac{1}{\eta}$

$$\Rightarrow \ddot\delta + 3\underbrace{\frac{\dot a}{a}}_{1/\eta}\dot\Phi + \underbrace{\frac{\dot a}{a}}_{1/\eta}\dot\delta - k^2\Phi = -3\ddot\Phi$$

$$\Rightarrow \boxed{\ddot\delta + \frac{1}{\eta}\dot\delta = S(k,\eta)} \quad \text{where } S(k,\eta) = -3\ddot\Phi + k^2\Phi - \frac{3}{\eta}\dot\Phi$$

$\ddot\delta + \frac{1}{\eta}\dot\delta = -3\ddot\Phi + k^2\Phi - \frac{3}{\eta}\dot\Phi$ equation for ~~DM~~ perturbations $\begin{cases} \cdot\ \text{RD} \\ \cdot\ \text{around horizon crossing} \end{cases}$

Homogeneous equation

$$\ddot\delta + \frac{1}{\eta}\dot\delta = 0 \quad \begin{cases} \delta = \text{const} \\ \delta = \ln(\eta) \quad [= \ln(a) \text{ during RD}] \end{cases}$$

$y = \dot\delta \quad \dot y + \frac{y}{\eta} = 0 \quad \frac{dy}{d\eta} = -\frac{y}{\eta} \quad \frac{dy}{y} = -\frac{d\eta}{\eta} \quad y \propto \frac{1}{\eta}$

$\delta = \text{const}$, $\delta = \ln(\eta)$ solutions to homogeneous equation
$\uparrow$ $s_1(\eta)$ $\uparrow$ $s_2(\eta)$

Recall initial condition $\delta(k,\eta_i) = \text{const} = \frac{3}{2}\Phi_p$

General solution to 2nd order equation

$\cancel{\delta(k,\eta) = C_1 C_2}$

$$\delta(k,\eta) = C_1 s_1(\eta) + C_2 s_2(\eta) + \int_0^\eta d\eta' \, S(k,\eta') \underbrace{\frac{s_1(\eta)s_2(\eta') - s_1(\eta')s_2(\eta)}{\dot{s}_1(\eta')s_2(\eta') - s_1(\eta')\dot{s}_2(\eta')}}_{G(\eta,\eta')}$$

$$G(\eta,\eta') = \frac{1 \times \ln(\eta') - 1 \times \ln(\eta)}{0 - 1/\eta'} = -\eta'\left[\ln(k\eta') - \ln(k\eta)\right]$$
$\uparrow$ $s_1(\eta) = 1$

$$\Rightarrow \delta(k,\eta) = C_1 + C_2 \ln(\eta) - \int_0^\eta d\eta' \, S(k,\eta')\eta'\left[\ln(k\eta') - \ln(k\eta)\right]$$

$$\xrightarrow[\eta \to 0]{} C_1 + C_2 \ln(\eta) = \frac{3}{2}\Phi_p \Rightarrow C_2 = 0, \quad C_1 = \frac{3\Phi_p}{2}$$

How does $S(k,\eta')$ behave?
When mode enters horizon $k\eta > 1$, Φ decays. For $k\eta < 1$ integral is small

$$S(k,\eta) = -3\ddot{\Phi} + k^2\Phi - \frac{3}{\eta}\dot{\Phi} \longrightarrow 0$$

Dominant contribution to integral $\int d\eta' S(k,\eta') G(\eta,\eta')$ when $k\eta \sim 1$

$$\int d\eta' S(k,\eta') G(\eta,\eta') \propto \underbrace{\int d\eta' S(\eta')\eta' \ln(k\eta')}_{\text{asymptote to constant}} + \underbrace{\int d\eta' S(\eta') \ln(k\eta)}_{\propto \ln(k\eta)}$$

So after mode entered horizon

$$\delta(k,\eta) \propto \ln(k\eta) = A\Phi_p \ln(Bk\eta) = \underbrace{A\Phi_p \ln(B)}_{\text{constant}} + \underbrace{A\Phi_p \ln(k\eta)}_{\text{logarithmically growing mode}}$$

$$\delta(k,\eta) = C_1 + \cancel{C_2 \ln(\eta)} + \int_0^\eta d\eta' \,\delta(k,\eta')\,\eta'\left(\ln(k\eta') - \ln(k\eta)\right)$$

$$= A\Phi_p \ln(B) + A\Phi_p \ln(k\eta)$$

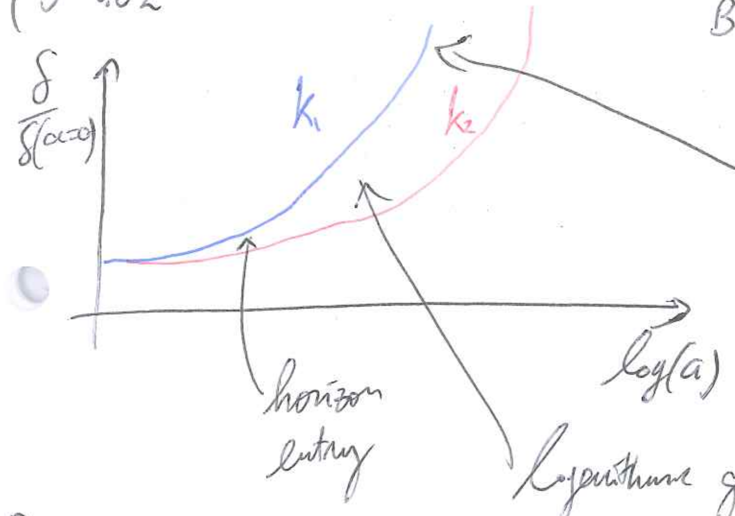
$$A\Phi_p \ln(B) = C_1 - \int_0^\infty d\eta' \,\delta(k,\eta')\,\eta' \ln(k\eta')$$

$$A\Phi_p = \int_0^\infty d\eta' \,\delta(k,\eta')\,\eta'$$

Upper limits of integration at ∞ since integrals asymptote to constant at large η

Knowing $\Phi \propto J_1(k\eta)$ we can do the integrals numerically

$\begin{cases} A = 9.0 \\ B = 0.62 \end{cases}$ $\xrightarrow{\text{integrating exact solution}}$ $A = 9.6$, $B = 0.44$

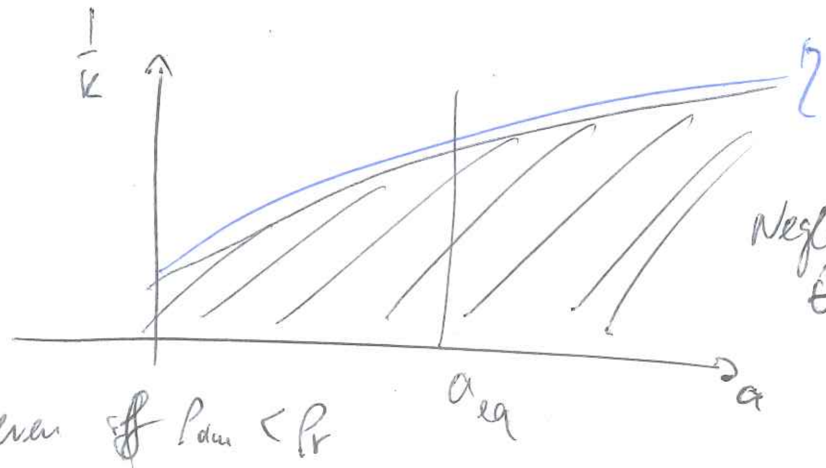


$k_1 > k_2$ (k_1 enters horizon earlier)

Recap: $\delta \propto \ln(\eta) \propto \ln(a)$ during RD $\quad\quad \delta \propto a$ during MD

Sub-horizon solution

Pressure also suppresses growth of Θ_r (while δ grows as $\delta \propto \ln\eta$)



Neglect Θ_r

At some point $\rho_{dm}\delta \gtrsim \rho_r \Theta_{r,0}$ even ~~if~~ $\rho_{dm} < \rho_r$

Once this happens, Φ and δ evolve together independent of radiation, so we can solve for Φ and δ matching on to logarithmic growing solution set up while Φ was decaying

$$\delta(k,\eta) = A\Phi_p \ln(Bk\eta)$$

Full set of equations (use algebraic Einstein equations)

$$\begin{cases} \dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi} \\ \dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi \\ \dot{\delta} + ikv = -3\dot{\Phi} \\ \dot{v} + \frac{\dot{a}}{a}v = ik\Phi \\ k^2\Phi = 4\pi G a^2\left[\rho_{dm}\delta + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}\left(i\rho_{dm}v + 4\rho_r\Theta_{r,1}\right)\right] \end{cases}$$

neglect radiation

$$\longrightarrow \begin{cases} \dot{\delta} + ikv = -3\dot{\Phi} \\ \dot{v} + \frac{\dot{a}}{a}v = ik\Phi \\ k^2\Phi \simeq 4\pi G a^2\left(\rho_{dm}\delta + \frac{3iaH}{k}\rho_{dm}v\right) \end{cases}$$

As usual we want to reduce these 3 equations to 1 2nd order equation for δ

We need to follow δ through equality, so again use variable y

Recall

$$y \equiv \frac{a}{a_{eq}} \qquad \text{ignoring baryons so} \qquad \frac{\rho_r}{\rho_{dm}} \simeq \frac{a_{eq}}{a} \simeq \frac{1}{y}$$

$$\dot{} = \frac{d}{d\eta} = \underset{\text{done earlier}}{\ldots} = aHy\frac{d}{dy} = aHy\,' \qquad \text{so} \qquad \dot{} \longrightarrow aHy\,'$$

$$\dot{\delta} + ikv = -3\dot{\Phi} \Rightarrow aHy\delta' + ikv = -3aHy\Phi' \Rightarrow \delta' + \frac{ikv}{aHy} = -3\Phi'$$

$$\dot{v} + \frac{\dot{a}}{a}v = ik\Phi \Rightarrow aHyv' + \frac{aHy}{a}a'v = ik\Phi$$

$$\frac{aHy}{a} a' v = Hya'v = \left\{ a' = \frac{da}{dy} = \left| y = \frac{a}{a_{eq}} \rightarrow dy = \frac{1}{a_{eq}} da \rightarrow \frac{d}{dy} = a_{eq} \frac{d}{da} \right. = a_{eq} \frac{da}{da} = a_{eq} \right.$$

$$= Hy a_{eq} v =$$

$$= H \frac{a}{a_{eq}} a_{eq} v = Hav$$

$$\Rightarrow aHy v' + aHv = ik\Phi \Longrightarrow v' + \frac{v}{y} = \frac{ik\Phi}{aHy}$$

$$k^2 \Phi \simeq 4\pi G a^2 \left(\rho_{dm} \delta + \frac{3iaH}{k} \rho_{dm} v\right) \quad \frac{aH}{k} \ll 1 \quad \simeq 4\pi G a^2 \rho_{dm} \delta =$$

Sub-horizon modes

Recall $y \simeq \frac{\rho_{dm}}{\rho_r} \Rightarrow y+1 \simeq \frac{\rho}{\rho_r} \Rightarrow \frac{y}{y+1} \rho \simeq \rho_{dm}$

$$= 4\pi G a^2 \rho \frac{y}{y+1} \delta = \cancel{4\pi G} a^2 \frac{3H^2}{\cancel{8\pi G}_{2}} \frac{y}{y+1} = \frac{3}{2} a^2 H^2 \frac{y}{y+1} \delta$$

$3H^2 = \frac{8\pi G}{3} \rho$

So final set of equations

$$\begin{cases} \delta' + \dfrac{ikv}{aHy} = -3\Phi' \\ v' + \dfrac{v}{y} = \dfrac{ik\Phi}{aHy} \\ k^2 \Phi \simeq \dfrac{3y}{2(y+1)} a^2 H^2 \delta \end{cases}$$

Now let's turn these into a 2nd order equation for δ

$$\frac{d}{dy}\left(\delta' + \frac{ikv}{aHy}\right) = \frac{d}{dy}(-3\Phi')$$

Useful: $\frac{d}{dy}\left(\frac{1}{aHy}\right) = -\frac{1}{2aHy(1+y)}$

Exercise!

$$\frac{d}{dy}\left(\delta' + \frac{ikv}{aHy}\right) = -3\Phi''$$

$$v' + \frac{v}{y} = \frac{ik\Phi}{aHy}$$

$$\frac{d}{dy}\left(\delta' + \frac{ikv}{aHy}\right) = \delta'' + \frac{ik}{aHy}v' + ikv\frac{d}{dy}\left(\frac{1}{aHy}\right) = \delta'' + \frac{ik}{aHy}\left(\frac{ik\Phi}{aHy} - \frac{v}{y}\right) + ikv\left(-\frac{1}{2aHy(1+y)}\right) =$$

$$= \delta'' - \frac{k^2\Phi}{a^2H^2y^2} - \frac{ik}{aHy^2}v - \frac{ikv}{2aHy(1+y)} = \delta'' - \frac{k^2\Phi}{a^2H^2y^2} - ikv\underbrace{\left(\frac{1}{aHy^2} + \frac{1}{2aHy(1+y)}\right)}_{\frac{2(1+y)+y}{2aHy^2(1+y)} = \frac{2+3y}{2aHy^2(1+y)}} =$$

$$= \delta'' - \frac{k^2\Phi}{a^2H^2y^2} - \frac{ik(2+3y)v}{2aHy^2(1+y)} = -3\Phi''$$

$$\Longrightarrow \delta'' - \frac{ik(2+3y)v}{2aHy^2(1+y)} = -3\Phi'' + \frac{k^2\Phi}{a^2H^2y} \overset{k \gg aH}{\simeq} \frac{k^2\Phi}{a^2H^2y^2} \overset{k^2\Phi = \frac{3y}{2(y+1)}a^2H^2\delta}{=} \frac{3\delta}{2y(y+1)} \quad *$$

$$\circledast \quad -\frac{ik(2+3y)}{2aHy^2(1+y)}v \simeq -\frac{ik(2+3y)}{2aHy^2(1+y)}\frac{iaHy}{k}(3\Phi' + \delta') \simeq -\frac{ik(2+3y)}{2aHy^2(1+y)}\frac{iaHy}{k}\delta' =$$

$$\delta' + \frac{ikv}{aHy} = -3\Phi' \Rightarrow v = \frac{iaHy}{k}(3\Phi' + \delta') \cancel{\simeq \frac{iaHy}{k}\delta \text{ since}}$$

since $\Phi \ll \delta$ on sub-horizon scales

$$= \frac{2+3y}{2y(1+y)}\delta'$$

$$* \quad \boxed{\delta'' + \frac{2+3y}{2y(1+y)}\delta' - \frac{3}{2y(y+1)}\delta = 0}$$

Meszaros equation for sub-horizon CDM perturbations with negligible radiation perturbations for $z \ll z_{eq}$

$$\delta(y) = \delta\left(\frac{a}{a_{eq}}\right)$$

To see how δ grows we need 2 independent solutions of the Meszaros equation, then match on to the logarithmic growth

$$\delta'' + \frac{2+3y}{2y(y+1)}\delta' - \frac{3}{2y(y+1)}\delta = 0$$

We know that deep during MD $\delta \propto a \propto y$, so one solution is a polynomial of order 1 in $y \rightarrow \delta \propto y \rightarrow \delta'' = 0$

For this ~~mode~~ growing mode, as long as $a \lesssim 0.1$ (afterwards DE is dominant so we need growth function)

$0 : D_1 \propto y$

$$\cancel{D_1''} + \frac{2+3y}{2y(y+1)} D_1' - \frac{3D_1}{2y(y+1)} = 0 \rightarrow \frac{2+3y}{\cancel{2y(y+1)}} D_1' = \frac{3D_1}{\cancel{2y(y+1)}}$$

$$\Rightarrow \boxed{\frac{D_1'}{D_1} = \frac{3}{2+3y}}$$ for the Meszaros growing mode

$$\frac{1}{D_1}\frac{dD_1}{dy} = \frac{3}{2+3y} \Rightarrow \int \frac{dD_1}{D_1} = \int \frac{3}{2+3y} dy \Rightarrow \ln D_1 = \ln(2+3y) + \text{const}$$

$\Rightarrow D_1(y) = 2+3y$ growing mode $\delta(y) = 2+3y \propto y$

To find the second solution consider $u = \frac{\delta}{D_1} = \frac{\delta}{y+\frac{2}{3}} = \frac{3}{2+3y}\delta$

$$\rightarrow \delta = \frac{2+3y}{3} u(y)$$

$$\delta' = u(y) + \frac{2+3y}{3} u'(y)$$

$$\delta'' = 2u'(y) + \frac{2+3y}{3} u''(y)$$

$$\delta'' + \frac{2+3y}{2y(y+1)}\delta' - \frac{3}{2y(y+1)}\delta = 2u' + \frac{2+3y}{3}u'' + \left(\frac{2+3y}{2y(y+1)}\right)\left[u + \frac{2+3y}{3}u'\right]$$

$$- \frac{\cancel{3}}{2y(y+1)}\frac{2+3y}{\cancel{3}}u =$$

$$= \frac{2+3y}{3}u'' + \frac{(2+3y)^2}{6y(y+1)}u' + 2u' + \cancel{\frac{2+3y}{2y(y+1)}u} - \cancel{\frac{2+3y}{2y(y+1)}u} =$$

$$= \frac{2+3y}{3}u'' + \frac{4+9y^2+12y+12y^2+12y}{6y(y+1)} = \frac{2+3y}{3}u'' + \frac{21y^2+24y+4}{6y(y+1)}u' = 0$$

Multiply by $\frac{3}{2}$

$$\Rightarrow \boxed{\left(1+\frac{3y}{2}\right)u'' + \frac{\left(\frac{21}{4}y^2+6y+1\right)}{y(y+1)}u' = 0}$$

No term proportional to u, so it is really a 1st order equation for u'

Call $x = u'$

$$\left(1+\frac{3y}{2}\right)x' + \frac{\left(\frac{21}{4}y^2+6y+1\right)}{y(y+1)}x = 0 \quad \rightarrow \quad \frac{dx}{x} = -\frac{\frac{21}{4}y^2+6y+1}{\left(1+\frac{3y}{2}\right)y(y+1)}dy$$

$$\int\frac{dx}{x} = -\int dy\,\frac{\frac{21}{4}y^2+6y+1}{\left(1+\frac{3y}{2}\right)y(y+1)} = -\ln y - \frac{1}{2}\ln(1+y) - 2\ln(2+3y)$$

$$\Rightarrow x = u' \propto \left(y+\frac{2}{3}\right)^{-2}y^{-1}(y+1)^{-1/2}$$

$$u(y) = \int dy\, u'(y) = \int dy\left[\left(y+\frac{2}{3}\right)^{-2}y^{-1}(y+1)^{-1/2}\right]$$

Integrating one gets

$$u \propto \frac{\sqrt{1+y}}{D_1(y)} - \operatorname{arctanh}\left(\sqrt{1+y}\right) = \frac{D_2(y)}{D_1(y)}$$

$$\Rightarrow D_2(y) = D_1(y) \ln\left[\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right] - 2\sqrt{1+y}$$

Late times $a \gg a_{eq}$, $y \gg 1$

$$D_1(y) = y + \frac{2}{3} \alpha y \qquad D_2(y) \propto y^{-3/2} \quad \text{decaying mode}$$

General solution for Meszaros equation

$$\delta(k,y) = C_1 D_1(y) + C_2 D_2(y) \quad \text{for } y > y_H$$

→ when the mode enters the horizon

C_1 and C_2 determined matching to earlier logarithmic solution valid for $y_H \ll y \ll 1$ (modes enter horizon before equality)

Match two solutions and their derivatives

- $\delta(k,\eta) = A\Phi_p \ln(Bk\eta) \simeq A\Phi_p \ln\left(B\frac{y}{y_H}\right)$ valid for $y \ll 1$
- $\delta(k,y) = C_1 D_1(y) + C_2 D_2(y)$

at matching time $y_H \ll y_m \ll 1$

$$A\Phi_p \ln\left(B\frac{y_m}{y_H}\right) = C_1 D_1(y_m) + C_2 D_2(y_m)$$

$$\frac{A\Phi_p}{y_m} = C_1 D_1'(y_m) + C_2 D_2'(y_m)$$

Summing up:

$\sim$ logarithmic growth + $\propto a$ growth

Summary

- Large-scale super-horizon solution (drop k terms)

Assumes $k\eta \ll 1$, mode crosses horizon deep in MD, so valid for $a \ll a_{eq}$, $a \sim a_{eq}$, $a \gg a_{eq}$

$\Phi \sim$ constant but drops by $\frac{1}{10}$ $\left(\Phi \to \frac{9}{10}\Phi\right)$ at matter-radiation equality even if mode is super-horizon

- Large-scale horizon crossing solution (Φ constant)

Valid for $a \gg a_{eq}$, $k\eta \ll 1$

$\Phi \sim$ Constant is also a solution
$\delta \propto a$

valid for $a \gg a_{eq}$

- Small-scale horizon crossing solution (neglect δ)

Valid for $a \ll a_{eq}$, $k\eta \gg 1$, mode crosses deep during RD.

$$\Phi = 3\Phi_p \left(\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3} \right)$$ decay + oscillations

$\delta(k,\eta) = A\Phi_p \ln(Bk\eta)$ logarithmic growth

- Small-scale sub-horizon solution (neglect Θ_r)

Valid for $a \sim a_{eq}$, $a \gg a_{eq}$, $k\eta \gg 1$

Meszaros equation

$\delta(y)$:

$D_1(y) = y + \frac{2}{3}$ $\quad y = \frac{a}{a_{eq}}$ $\quad D_1(a) \propto a$

$D_2(y) = D_1(y) \ln\left[\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right] - 2\sqrt{1+y}$ $\quad \propto y^{-3/2}$ (late times)

Now we can explore them...

Numerical results and fits

- Splining everything together we can get the transfer function

On small scales $\delta(k,y) = C_1 D_1(y) + C_2 D_2(y)$ can find C_1 explicitly (see Dodelson)

$$\rightarrow \delta(k,a) = \frac{3A\Phi_p(k)}{2} \ln\left[\frac{4B e^{-3} a_{eq}}{a_H}\right] D_1(a)$$

where $a_H H = K$, a_H scale factor when k crosses horizon

$$\frac{a_{eq}}{a_H} = \frac{\sqrt{2}k}{k_{eq}}$$

Recalling definition of transfer function

$$\delta(k,a) = \frac{3}{5}\frac{k^2}{\Omega_m H_0^2} \Phi_p(k) T(k) D_1(a)$$

$$\Rightarrow T_{\delta}(k) = \frac{5\,\delta(k,a)}{3k^2}\frac{\Omega_m H_0^2}{\Phi_p(k) D_1(a)} = \frac{5}{2}\frac{A\Omega_m H_0^2}{k^2 a_{eq}} \ln\left[\frac{4Be^{-3}\sqrt{2}\,k}{k_{eq}}\right]$$

for $k \gg k_{eq}$

$$\boxed{T(k) \simeq 12\frac{k_{eq}^2}{k^2}\ln\left[\frac{k}{8k_{eq}}\right]} \quad \text{for } k \gg k_{eq}$$

So on small scales $T(k) \propto \frac{\ln k}{k^2}$

- and $P(k) \propto P_{prim}(k) T^2(k) \propto k^{n_s} T^2(k) \propto \frac{\ln k}{k^{4-n}} \propto \frac{\ln k}{k^3}$ for $n_s \sim 1$

on large scales $P(k) \propto k^{n_s} \sim k$

Growth function

At late times ($z \lesssim 10$) all modes we care about entered horizon.

But we can use ~~St~~ $y \gg 1$ limit of Meszaros equation only if $\Omega_m = 1$, in which case all modes experience the same growth factor (k does not enter). When DE dominates we need to generalize this

We started from: (for $\Omega_m = 1$)

$$\begin{cases} \delta' + \dfrac{ikv}{aHy} = -3\Phi' \\ v' + \dfrac{v}{y} = \dfrac{ik\Phi}{aHy} \\ k^2\Phi = \dfrac{3y}{2(y+1)} a^2 H^2 \delta \end{cases}$$

$\downarrow$ Exercise ($y \to a$, neglect radiation)

$$\boxed{\frac{d^2\delta}{da^2} + \left(\frac{d\ln(H)}{da} + \frac{3}{a}\right)\frac{d\delta}{da} - \frac{3\Omega_m}{2a^2}\frac{H_0^2}{H}\delta = 0}$$

Two solutions:
- $\delta \propto H$ but this is decaying since $\dfrac{dH}{da} < 0$
- growing mode

Try $u \equiv \dfrac{\delta}{H}$

$$\hookrightarrow \frac{d^2u}{da^2} + 3\left[\frac{d\ln(H)}{da} + \frac{1}{a}\right]\frac{du}{da} = 0$$

Integrate: $v \equiv \dfrac{du}{da}$ $\quad \dfrac{dv}{da} + 3\left(\dfrac{d\ln(H)}{da} + \dfrac{1}{a}\right)v = 0$

$$\Rightarrow \frac{dv}{v} = -3\int \frac{d\ln(H)}{da}\, da - 3\int \frac{da}{a} = -3\ln H - 3\ln a = \ln\left((aH)^{-3}\right)$$

$$\Rightarrow \frac{du}{da} \propto (aH)^{-3} \Longrightarrow u(a) \propto \int_0^a \frac{da'}{(a'H(a'))^3} = \frac{\delta}{H}$$

$$\Rightarrow \boxed{\delta \propto D_1(a) \propto H(a)\int_0^a \frac{da'}{[a'H(a')]^3}}$$

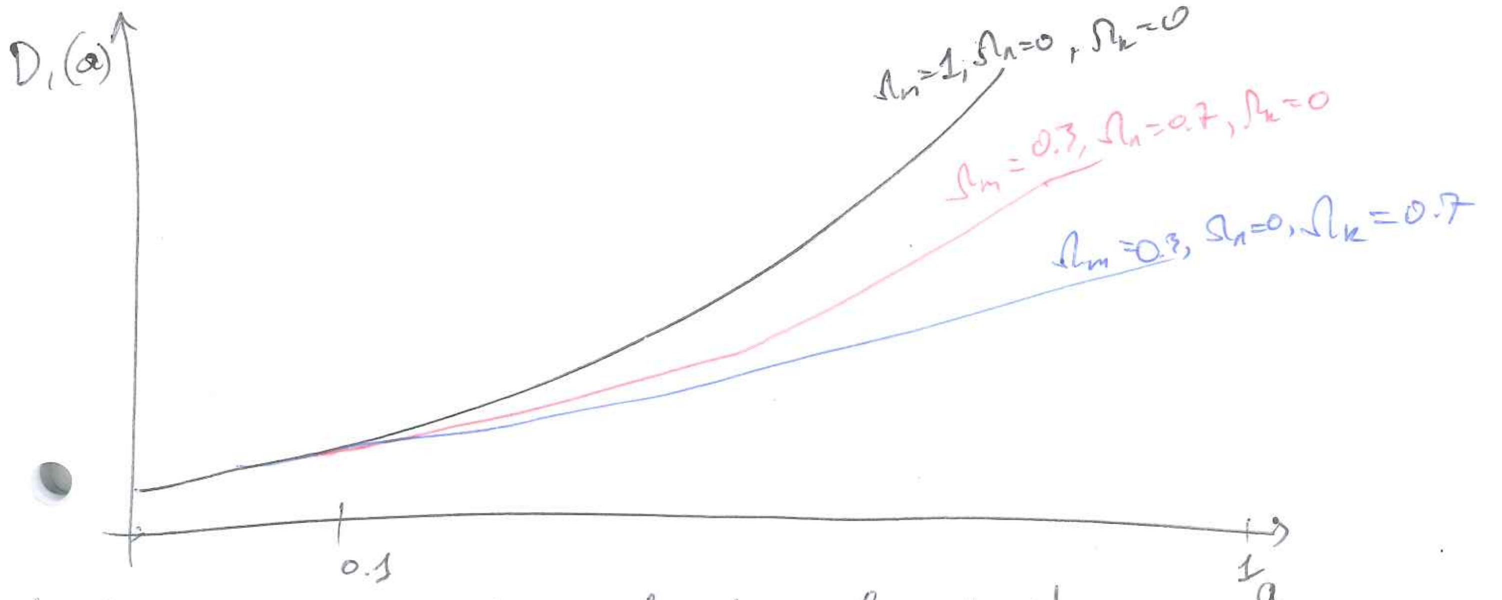
Fixing the proportionality constant

$$D_1(a) = \frac{5\Omega_m}{2}\frac{H(a)}{H_0}\int_0^a \frac{da'}{\left(\frac{a'H(a')}{H_0}\right)^3}$$

Check flat, MD Universe

$$\delta \propto H(a)\int_0^a \frac{da'}{[a'H(a')]^3} \propto a^{-3/2}\int \frac{da'}{[a'\,a'^{-3/2}]^3} \propto a^{-3/2}\int \frac{da'}{a'^{-3/2}}$$

$$\propto a^{-3/2} a^{5/2} \propto a \qquad \delta \propto a \text{ as we knew}$$



Dark energy slows down structure formation!

Beyond cold DM

Here we approximated $m \simeq DM$. But there is also:

- baryons
  - → suppress $P(k)$ on small scales (pressure) → b-γ coupling
  - → acoustic oscillations
- massive neutrinos
  - → power suppression on small scales
- Dark energy
  - → turnover in $P(k)$ indirect effect due to lower Ω_m
  - → $\delta \propto \dfrac{\Phi}{\Omega_m}$ raises $P(k)$ at Φ fixed
  - → Growth of $\delta \propto D(a)$ affected

Note: we don't measure $P_m(k)$, but $P_t(k)$

"tracer"

→ galaxies, QSOs, Ly-α ...

On large scales $P_t(k) \simeq \underbrace{b}_{\text{"bias"}} P_m(k)$