Acoustic oscillations

For $\eta \ll \eta_*$, electrons ionized, $\lambda_{MFP} \ll H^{-1}$, γ-b tightly coupled

Tightly coupled limit: $\tau \gg 1$ $\quad \left[\tau \equiv \int_\eta^{\eta_0} d\eta' a n_e \sigma_T\right]$

In $\tau \gg 1$ limit only non-negligible moments are Θ_0 and Θ_1

(so γ are a fluid just described by ρ and $\bar{v}$)

Consider Boltzmann equation for γ

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_e - \frac{1}{2}P_2(\mu)\Pi\right]$$

Turn it into infinite set of coupled equations for Θ_ℓ

Recall

$$\Theta_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)$$

Neglect polarization

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_b\right]$$

Consider $l \geq 2$ moments, multiply by $P_\ell(\mu)$, integrate over μ

$$\int\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu) = (-i)^\ell\Theta_\ell \qquad \int d\mu\, P_\ell(\mu)\Theta(\mu) = 2(-i)^\ell\Theta_\ell$$

$$\boxed{\frac{1}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\dot{\Theta}(\mu)} + \boxed{\frac{1}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\,ik\mu\Theta(\mu)} = \boxed{\frac{\dot{\tau}}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)} \qquad \boxed{\ell \geq 2}$$

$$\boxed{-\frac{1}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\dot{\Phi} - \frac{ik}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\Psi - \frac{\dot{\tau}}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\Theta_0 - \frac{\dot{\tau}v_b}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\mu}$$

$\boxed{}$: terms all of the form

$$\cancel{\int} \ \underbrace{\propto \int d\mu \, P_\ell(\mu)\,\mu}_{\propto \int d\mu \, P_{\ell \geq 2}(\mu) P_1(\mu) \ \propto \ \delta_{\ell \geq 2, 1} = 0} \quad \text{or} \quad \underbrace{\int d\mu \, P_\ell(\mu)}_{\propto \int d\mu \, P_{\ell \geq 2}(\mu) P_0(\mu) \ \propto \ \delta_{\ell \geq 2, 0} = 0}$$

So all $\boxed{}$ terms $= 0$

$$\frac{1}{(-i)^\ell} \int \frac{d\mu}{2} P_\ell(\mu) \dot{\Theta}(\mu) = \frac{1}{(-i)^\ell} \int \frac{d\mu}{2} P_\ell(\mu) \frac{\partial \Theta}{\partial \eta}(\mu) = \frac{\partial}{\partial \eta}\left[\frac{1}{(-i)^\ell} \int \frac{d\mu}{2} P_\ell(\mu) \Theta(\mu)\right] =$$

$$= \frac{\partial}{\partial \eta} \Theta_\ell = \dot{\Theta}_\ell$$

$$\frac{\dot{\tau}}{(-i)^\ell} \int \frac{d\mu}{2} P_\ell(\mu) \Theta(\mu) = \dot{\tau} \Theta_\ell$$

$$\frac{1}{(-i)^\ell} \int \frac{d\mu}{2} P_\ell(\mu) i k \mu \Theta(\mu) = \frac{k}{(-i)^{\ell+1}} \int \frac{d\mu}{2} P_\ell(\mu) \mu \Theta(\mu)$$

Use recurrence relation for Legendre polynomials

$$(\ell+1) P_{\ell+1}(\mu) = (2\ell+1) \mu P_\ell(\mu) - \ell P_{\ell-1}(\mu)$$

$$\Rightarrow \mu P_\ell(\mu) \Theta(\mu) = \frac{\ell+1}{2\ell+1} P_{\ell+1}(\mu) + \frac{\ell}{2\ell+1} P_{\ell-1}(\mu)$$

$$\frac{1}{(-i)^{\ell+1}} = \frac{1}{(-i)^{\ell-1}} \frac{1}{(-i)^2}$$

$$\Rightarrow \frac{k}{(-i)^{\ell+1}} \int \frac{d\mu}{2} \mu P_\ell(\mu) \Theta(\mu) = \frac{k(\ell+1)}{(2\ell+1)} \underbrace{\boxed{\frac{1}{(-i)^{\ell+1}} \int \frac{d\mu}{2} P_{\ell+1}(\mu) \Theta(\mu)}}_{\Theta_{\ell+1}} + \frac{k\ell}{(2\ell+1)} \underbrace{\boxed{\frac{1}{(-i)^{\ell+1}} \int \frac{d\mu}{2} P_{\ell-1}(\mu) \Theta(\mu)}}_{\Theta_{\ell-1} \, / \, \boxed{(-i)^2}} =$$

$$= \frac{k(\ell+1)}{2\ell+1} \Theta_{\ell+1} + \frac{k\ell}{(2\ell+1)} \frac{1}{(-i)^2} \Theta_{\ell-1} = \frac{k(\ell+1)}{2\ell+1} \Theta_{\ell+1} - \frac{k\ell}{2\ell+1} \Theta_{\ell-1}$$

Summarizing integrated Boltzmann equation ($\ell > 2$)

$$\boxed{\dot{\Theta}_\ell - \frac{k\ell}{2\ell+1}\Theta_{\ell-1} + \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1} = \dot{\tau}\Theta_\ell}$$

Look at orders of magnitude (for the moment neglect $\Theta_{\ell+1}$)

$$\dot{\Theta}_\ell \sim \frac{\Theta_\ell}{\eta} \ll \dot{\tau}\Theta_\ell \sim \tau\frac{\Theta_\ell}{\eta} \qquad \text{since } \tau \gg 1$$

$$\frac{k\ell}{2\ell+1}\Theta_{\ell-1} \sim \frac{k}{2}\Theta_{\ell-1}$$

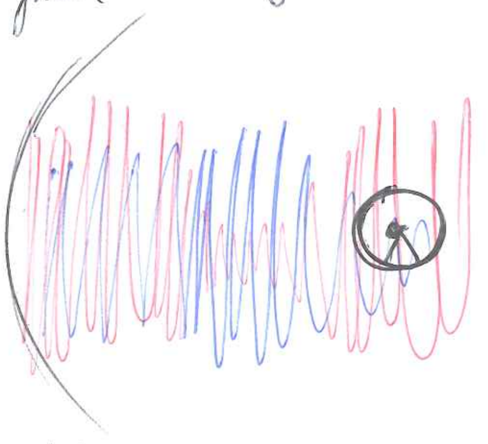
so $\dot{\Theta}_\ell - \frac{k\ell}{2\ell+1}\Theta_{\ell-1} \sim -\frac{k}{2}\Theta_{\ell-1} \sim \dot{\tau}\Theta_\ell \sim \frac{\tau\Theta_\ell}{\eta} \Rightarrow \frac{k}{2}\Theta_{\ell-1} \sim \frac{\tau\Theta_\ell}{\eta}$

$$\Rightarrow \boxed{\Theta_\ell \sim \frac{k\eta}{2\tau}\Theta_{\ell-1}}$$

For horizon-sized modes $k\eta \sim 1 \Rightarrow \boxed{\Theta_\ell \ll \Theta_{\ell-1}}$

So all $\ell > 2$ modes are $\ll \Theta_0, \Theta_1$ $\to$ can neglect all modes except monopole and dipole $\to$ fluid approximation

Physical meaning



observer sees photons arriving from $\lambda_{MFP} \sim \frac{\eta}{\tau}$, for $k\eta \sim 1$ perturbation wavelength much larger, so see photons with same temperature. For small wavelength $k\eta \sim \tau$, this is smaller than the damping scale, so only monopole and dipole survive

Summary: in the tightly coupled regime, only Θ_0 and Θ_1 survive

So now get equations for Θ_0 and Θ_1 starting from:

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_b\right] \quad \text{with} \quad \begin{cases} \Theta_0 = \int \frac{d\mu}{2}\,\Theta(\mu) \\ \Theta_1 = i\int \frac{d\mu}{2}\,\mu\,\Theta(\mu) \\ \Theta_2 = 0 \to \int d\mu\,\mu^2\Theta \simeq \frac{\Theta_0}{3} \end{cases}$$

Monopole

$$\underbrace{\int\frac{d\mu}{2}\dot{\Theta}}_{\dot{\Theta}_0} + ik\underbrace{\int\frac{d\mu}{2}\mu\Theta}_{\frac{\Theta_1}{i}} = -\dot{\Phi}\underbrace{\int\frac{d\mu}{2}}_{\frac{\mu}{2}\big|_{-1}^{1}=1} - ik\Psi\underbrace{\int\frac{d\mu}{2}\mu}_{\frac{\mu^2}{4}\big|_{-1}^{1}=0} - \dot{\tau}\Theta_0\underbrace{\int\frac{d\mu}{2}}_{\frac{\mu}{2}\big|_{-1}^{1}=1} + \dot{\tau}\underbrace{\int\frac{d\mu}{2}\Theta}_{\Theta_0} - \dot{\tau}v_b\underbrace{\int\frac{d\mu}{2}\mu}_{\frac{\mu^2}{4}\big|_{-1}^{1}=0}$$

$$\Rightarrow \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} - \cancel{\dot{\tau}\Theta_0} + \cancel{\dot{\tau}\Theta_0} \Rightarrow \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi}$$

Dipole

$$i\underbrace{\int\frac{d\mu}{2}\mu\dot{\Theta}}_{\dot{\Theta}_1} - k\underbrace{\int\frac{d\mu}{2}\mu^2\Theta}_{\simeq\frac{\Theta_0}{3}} = -i\dot{\Phi}\underbrace{\int\frac{d\mu}{2}\mu}_{\frac{\mu^2}{4}\big|_{-1}^{1}=0} + k\Psi\underbrace{\int\frac{d\mu}{2}\mu^2}_{\frac{\mu^3}{6}\big|_{-1}^{1}=\frac{1}{3}} - i\dot{\tau}\Theta_0\underbrace{\int\frac{d\mu}{2}\mu}_{\frac{\mu^2}{4}\big|_{-1}^{1}=0} + i\dot{\tau}\underbrace{\int\frac{d\mu}{2}\mu\Theta}_{\dot{\tau}\Theta_1} - i\dot{\tau}v_b\underbrace{\int\frac{d\mu}{2}\mu^2}_{\frac{\mu^3}{6}\big|_{-1}^{1}=\frac{1}{3}}$$

$$\Rightarrow \dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\Psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{iv_b}{3}\right]$$

So equations for monopole and dipole in tightly-coupled limit:

$$\boxed{\begin{aligned} &\dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \\ &\dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\Psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{iv_b}{3}\right] \end{aligned}}$$

Need also equations for baryon fluid
$$\begin{cases} \dot{\delta}_b + ikv_b = -3\dot{\Phi} \\ \dot{v}_b + \frac{\dot{a}}{a}v_b = -ik\Psi + \frac{\dot{\tau}}{R}\left[v_b + 3i\Theta_1\right] \end{cases}$$

Consider velocity equation

$$\dot{v}_b + \frac{\dot{a}}{a} v_b = -ik\psi + \frac{\dot{\tau}}{R}\left[v_b + 3i\Theta_1\right]$$

Rewrite it as $\rightarrow$ $$v_b = -3i\Theta_1 + \underbrace{\frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a}v_b + ik\psi\right]}_{\text{Suppressed by } \tau^{-1} \ll 1 \text{ relative to } -3i\Theta_1}$$

So to lowest order $v_b \simeq -3i\Theta_1$ (i.e. $v_b \approx v_\gamma$) and we can use this lowest-order expansion everywhere

$$v_b = -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a}v_b + ik\psi\right] \longrightarrow -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[3i\dot{\Theta}_1 - \frac{\dot{a}}{a}3i\Theta_1 + ik\psi\right]$$

$$\rightarrow v_b \simeq -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[-3i\dot{\Theta}_1 - 3i\frac{\dot{a}}{a}\Theta_1 + ik\psi\right]$$

$\hookrightarrow$ plug into dipole equation $$\dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{iv_b}{3}\right]$$

$$\Theta_1 - \frac{iv_b}{3} \simeq \Theta_1 - \Theta_1 - \frac{i}{3}\frac{R}{\dot{\tau}}\left[-3i\dot{\Theta}_1 - 3i\frac{\dot{a}}{a}\Theta_1 + ik\psi\right] = \cancel{\Theta_1} - \cancel{\Theta_1} - \frac{R}{\dot{\tau}}\left[\dot{\Theta}_1 + \frac{\dot{a}}{a}\Theta_1 - \frac{k\psi}{3}\right] =$$

$$= -\frac{R}{\dot{\tau}}\left[\dot{\Theta}_1 + \frac{\dot{a}}{a}\Theta_1 - \frac{k\psi}{3}\right]$$

$$\dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{iv_b}{3}\right] \Rightarrow \dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\psi}{3} - \dot{\tau}\frac{R}{\dot{\tau}}\left[\dot{\Theta}_1 + \frac{\dot{a}}{a}\Theta_1 - \frac{k\psi}{3}\right] =$$

$$= -R\dot{\Theta}_1 - \frac{\dot{a}}{a}R\Theta_1 + \frac{k\psi}{3}(1+R)$$

$$\Rightarrow \dot{\Theta}_1 - \frac{k\Theta_0}{3} = -R\dot{\Theta}_1 - \frac{\dot{a}}{a}R\Theta_1 + \frac{k\psi}{3}(1+R)$$

$$\Rightarrow \dot{\Theta}_1(1+R) + \frac{\dot{a}}{a}R\Theta_1 - \frac{k\Theta_0}{3} = \frac{k\psi}{3}(1+R)$$

$$\Rightarrow \boxed{\dot{\Theta}_1 + \frac{\dot{a}}{a}\frac{R}{1+R}\Theta_1 - \frac{k}{3(1+R)}\Theta_0 = \frac{k\Psi}{3}}$$

So now we have two coupled 1st order equations for Θ_0 and Θ_1

$$\begin{cases} \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} & (1) \\ \dot{\Theta}_1 + \frac{\dot{a}}{a}\frac{R}{1+R}\Theta_1 - \frac{k}{3(1+R)}\Theta_0 = \frac{k\Psi}{3} & (2) \end{cases}$$

Turn this into 2nd order equation for Θ_0 by differentiating (1), using (2) to eliminate $\dot{\Theta}_1$

$$\ddot{\Theta}_0 + k\dot{\Theta}_1 = -\ddot{\Phi} \quad \text{with} \quad \dot{\Theta}_1 = -\frac{\dot{a}}{a}\frac{R}{1+R}\Theta_1 + \frac{k}{3(1+R)}\Theta_0 + \frac{k\Psi}{3}$$

$$\Rightarrow \ddot{\Theta}_0 + k\left(-\frac{\dot{a}}{a}\frac{R}{1+R}\Theta_1 + \frac{k}{3(1+R)}\Theta_0 + \frac{k\Psi}{3}\right) = -\ddot{\Phi}$$

$$\Rightarrow \cancel{\ddot{\Theta}_0 + k \frac{k\Psi}{3}} \quad \text{then use: } \Theta_1 = -\frac{\dot{\Phi}}{k} - \frac{\dot{\Theta}_0}{k}$$

$$\ddot{\Theta}_0 + k\left[-\frac{\dot{a}}{a}\left(-\frac{\dot{\Phi}}{k} - \frac{\dot{\Theta}_0}{k}\right)\frac{R}{1+R} + \frac{k}{3(1+R)}\Theta_0 + \frac{k\Psi}{3}\right] = -\ddot{\Phi}$$

$$\Rightarrow \ddot{\Theta}_0 + \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0 + \frac{k^2}{3(1+R)}\Theta_0 = -\frac{k^2\Psi}{3} - \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Phi} - \ddot{\Phi}$$

Define sound speed of the fluid as

$$c_s \equiv \sqrt{\frac{1}{3(1+R)}} = \sqrt{\frac{1}{3\left(1+\frac{3\rho_b}{4\rho_\gamma}\right)}} \quad \begin{matrix} \xrightarrow[a\to 0]{\rho_b \to 0} \frac{1}{\sqrt{3}} \\ \xrightarrow[a\to\infty]{\rho_\gamma\to 0} 0 \end{matrix}$$

$$R \equiv \frac{3\rho_b}{4\rho_\gamma}$$

With this definition $c_s \downarrow$ as $R \uparrow$ because baryons make fluid heavier

$$\boxed{\ddot{\Theta}_0 + \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0 + k^2 c_s^2 \Theta_0 = -\frac{k^2}{3}\Psi - \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Phi} - \ddot{\Phi} \equiv F(k,\eta)}$$

where driving force

$$F(k,\eta) = -\frac{k^2}{3}\Psi - \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Phi} - \ddot{\Phi}$$

This is the more complete version of the cartoon equation

$$\ddot{\Theta}_0 + k^2 c_s^2 \Theta_0 = F$$

by additional damping term $\frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0$

$$\boxed{\ddot{\Theta}_0 + \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0 + k^2 c_s^2 \Theta_0 = F(k,\eta)}$$

Note Φ enters in the right in a way very similar to Θ_0 on the left

Alternate form of equation

$$\boxed{\left\{\frac{d^2}{d\eta^2} + \frac{\dot{R}}{1+R}\frac{d}{d\eta} + k^2 c_s^2\right\}\left[\Theta_0 + \Phi\right] = \frac{k^2}{3}\left[\frac{1}{1+R}\Phi - \Psi\right]}$$

Tightly coupled solutions

$\boxed{\quad}$ = 2nd order ODE, need 2 solutions to homogeneous equation to construct Green's function

To get some insight neglect damping

damping term $\simeq \frac{\dot{R}}{1+R}\frac{d}{d\eta}(\Theta_0 + \Phi) \sim \frac{R(\Theta_0+\Phi)}{\eta^2}$

pressure term $\sim k^2 c_s^2 (\Theta_0 + \Phi)$

$\frac{\text{pressure}}{\text{damping}} \sim \frac{k^2 c_s^2 \eta^2}{R} \gg 1$ if

R small

modes within horizon $k^2\eta^2 \gg 1$

We expect pressure to set up oscillations on scale much smaller than damping due to expansion of Universe

- Approximate (oscillating) equation

$$\left(\frac{d^2}{d\eta^2}+k^2c_s^2\right)\left[\Theta_0+\Phi\right]=\frac{k^2}{3}\left[\frac{\Phi}{1+R}-\Psi\right]$$

Homogeneous equation

$$\left[\frac{d^2}{d\eta^2}+k^2c_s^2(\eta)\right](\Theta_0+\Phi)=0$$

- solutions to the homogeneous equation

$$S_1(k,\eta)=\sin\left[kr_s(\eta)\right] \qquad S_2(k,\eta)=\cos\left[kr_s(\eta)\right]$$

Sound horizon $\quad r_s(\eta)\equiv\int_0^\eta d\eta' c_s(\eta')$

Now can construct full solution. In source term drop R $(1+R\sim 1)$ so it's just $\frac{k^2}{3}(\Phi-\Psi)$, but don't drop R in sin/cos argument

- $$\Theta_0(\eta)+\Phi(\eta)\simeq C_1S_1(\eta)+C_2S_2(\eta)+\frac{k^2}{3}\int_0^\eta d\eta'\left[\Phi(\eta')-\Psi(\eta')\right]\frac{\boxed{S_1(\eta')S_2(\eta)-S_1(\eta)S_2(\eta')}}{\boxed{S_1(\eta')\dot{S}_2(\eta')-\dot{S}_1(\eta')S_2(\eta')}}$$

$$S_1(\eta')S_2(\eta)-S_1(\eta)S_2(\eta')=\sin\left[kr_s(\eta')\right]\cos\left[kr_s(\eta)\right]-\sin\left[kr_s(\eta)\right]\cos\left[kr_s(\eta')\right]=$$

$$=\sin\left[k(r_s(\eta)-r_s(\eta'))\right]$$ (just a simple trigonometric identity)

$$S_1(\eta')\dot{S}_2(\eta')-\dot{S}_1(\eta')S_2(\eta')=?$$

$$\dot{S}_1(\eta')=+k\frac{dr_s}{d\eta}\cos\left[kr_s(\eta')\right] \qquad \frac{dr_s}{d\eta}=\frac{d}{d\eta}\int_0^\eta d\eta' c_s(\eta')=c_s \longrightarrow =+kc_s\cos\left[kr_s(\eta')\right]$$

- and similarly $\quad \dot{S}_2(\eta')=-kc_s\sin\left[kr_s(\eta')\right]$

So $S_1(\eta')\dot{S}_2(\eta')-\dot{S}_1(\eta')S_2(\eta')=-kc_s(\eta')\sin^2\left[kr_s(\eta')\right]-kc_s(\eta')\cos^2\left[kr_s(\eta')\right]=-kc_s(\eta')$

but in the limit we are working in

$R \ll 1 \Rightarrow 1+R \sim 1 \Rightarrow c_s \simeq \frac{1}{\sqrt{3}}$

$\Rightarrow -k c_s(\eta') \simeq -\frac{k}{\sqrt{3}}$

Putting everything together

$$\frac{S_1(\eta')S_2(\eta) - S_1(\eta)S_2(\eta')}{S_1(\eta')\dot{S}_2(\eta') - \dot{S}_1(\eta')S_2(\eta')} \simeq \frac{-\sin[k(r_s(\eta)-r_s(\eta'))]}{-k/\sqrt{3}} = \frac{\sqrt{3}\sin[k(r_s(\eta)-r_s(\eta'))]}{k}$$

$$\Theta_0(\eta)+\Phi(\eta) \simeq C_1 \sin[k r_s(\eta)] + C_2 \cos[k r_s(\eta)]$$
$$+ \frac{k}{\sqrt{3}}\int_0^{\eta} d\eta' \, [\Phi(\eta')-\Psi(\eta')] \sin[k(r_s(\eta)-r_s(\eta'))]$$

Now just need to fix C_1 and C_2. Initial conditions:

$\Theta_0(k,\eta_*) = \frac{2\Phi_p}{3} = \text{const}$

$\Rightarrow C_1 = 0 \; , \; C_2 = \Theta_0(0)+\Phi(0)$ as $\sin[k r_s(\eta)] \xrightarrow{\eta\to 0} 0$

$\cos[k r_s(\eta)] \xrightarrow{\eta\to 0} 1$

$\Rightarrow$ Full expression of anisotropy in the tightly coupled limit

$$\boxed{\Theta_0(\eta)+\Phi(\eta) = [\Theta_0(0)+\Phi(0)]\cos(k r_s(\eta)) + \frac{k}{\sqrt{3}}\int_0^{\eta} d\eta' \, \underbrace{[\Phi(\eta')-\Psi(\eta')]} \sin[k(r_s(\eta)-r_s(\eta'))]}$$

we get it from elsewhere (e.g. previous chapter)

Noteworthy feature of solution:

- agrees well with full numerical solution for location of acoustic peaks and height of first (undamped) peaks
- clearly separates solving for Φ, Ψ, then computing their effect on anisotropies
- shows that inflation excites cosine mode (without inflation by causality should have no perturbations for $k\eta \ll 1$ and therefore sine mode should be excited)
- when cos term dominates, peaks at

$$\cos(k_p r_s) = \pm 1 \longrightarrow k_p r_s = n\pi \longrightarrow k_p = \frac{n\pi}{r_s}$$

- Reduced full set of Θ_ℓ equations to just one!

Beyond Θ_0, also Θ_1 important. Recall

$$\dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \longrightarrow \Theta_1 = -\frac{\dot{\Phi}}{k} - \frac{\dot{\Theta}_0}{k} \quad \text{involves } \dot{\Theta}_0$$

$\cos \rightarrow -\sin$

$$\Theta_1(\eta) \simeq \boxed{\frac{1}{\sqrt{3}}\left[\Theta_0(0) + \Phi(0)\right]\sin\left[k r_s(\eta)\right]} - \frac{k}{3}\int_0^\eta d\eta' \left[\Phi(\eta') - \Psi(\eta')\right]\cos\left[k\left(r_s(\eta) - r_s(\eta')\right)\right]$$

completely out of phase with Θ_0 [$\sin(kr_s)$ versus $\cos(kr_s)$!]