ANISOTROPIES

- Previously studied matter inhomogeneities (δ), now care about photon anisotropies (CMB)

Overview

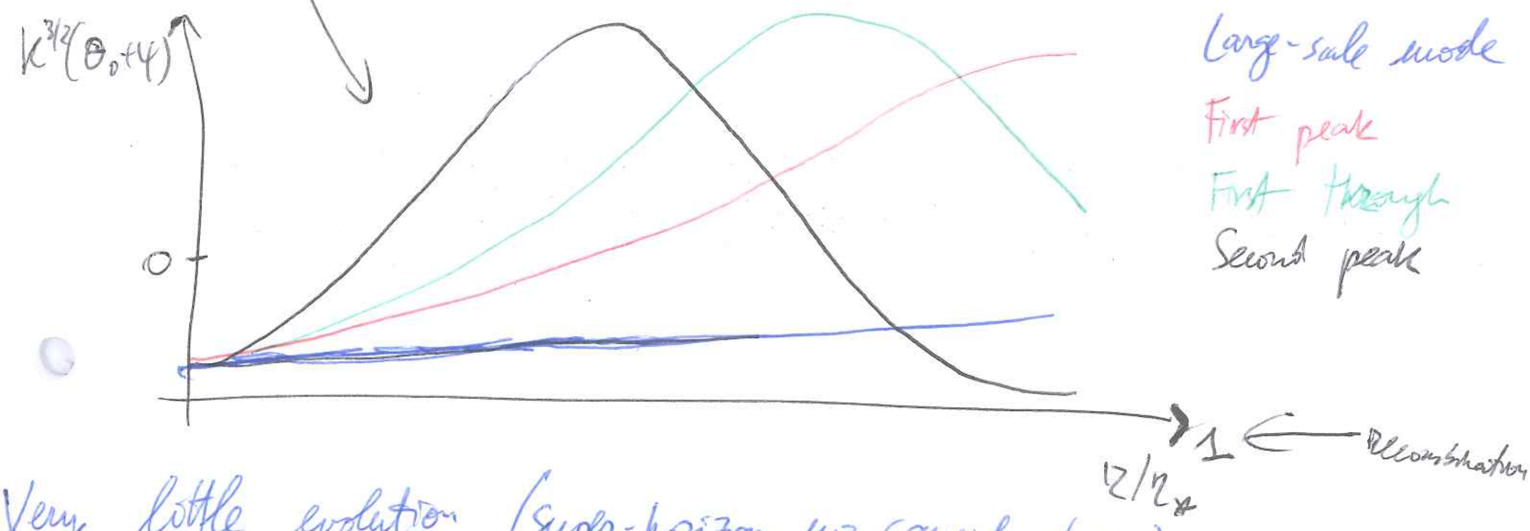
A posteriori, look at $K^{3/2}(\Theta_0 + \Psi)$

Amplitude of perturbations $\propto K^{-3/2}$

Ψ accounts for redshift when climbing out of overdensities ($\Psi<0$) or blueshift when entering underdensities ($\Psi>0$)

So we really observe $\Theta_0 + \Psi$

General feature: photon perturbations grow very little, remain quite linear (unlike δ)



Very little evolution (super-horizon, no causal physics)

Reaches maximum at recombination → expect large fluctuations on corresponding scales (for anisotropies)

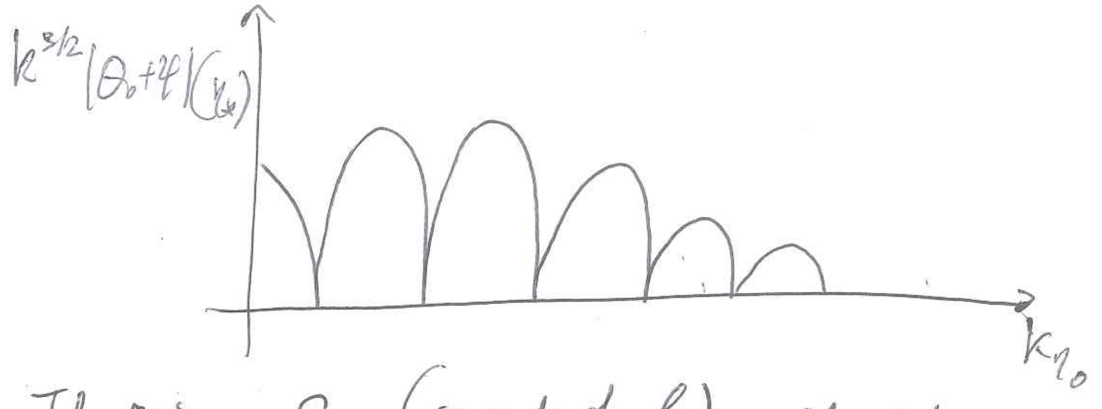
Amplitude at recombination zero (has done exactly half oscillation) expect small fluctuations

- One full oscillation by recombination

→ signatures of acoustic oscillations

So we expect a series of peaks and troughs in CMB anisotropy spectrum for smaller and smaller scales whose modes entered horizon earlier

Example: $k^{3/2}|\Theta_0+\Psi|(\eta_*)$ vs $k\eta_0$ (snapshot at recombination)



If raise Ω_b (amount of b) odd peaks are higher than even peaks, and small-scale perturbations ($k\eta_0 \gtrsim 500$) are damped

Cartoon version of equations

$$\ddot{\Theta}_0 + k^2 c_s^2 \Theta_0 = F$$

pressure (restoring force)

~~restoring~~ driving force

Forced harmonic oscillator

Review of forced harmonic oscillator

$$F = F_0 - kx \longrightarrow \ddot{x} + \frac{k}{m}x = +\frac{F_0}{m}$$

drives to large x

restores to small x ($x \to 0$)

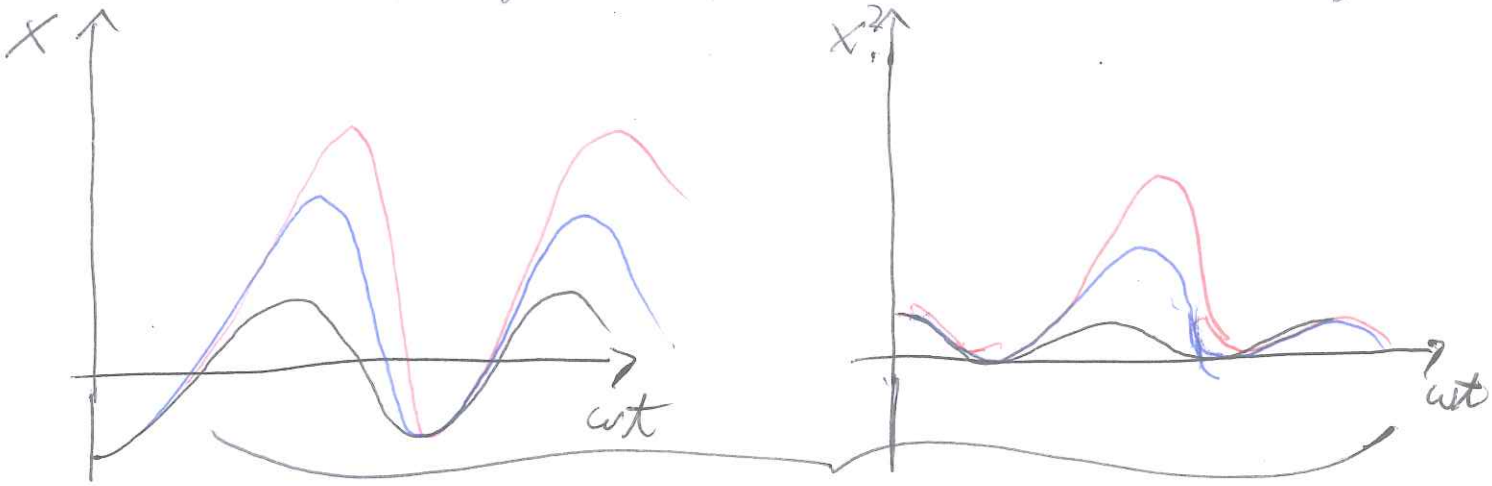
Full solution (if oscillator initially at rest)

$$x = A\cos(\omega t) + \frac{F_0}{m\omega^2} \qquad \omega^2 = \frac{k}{m}$$

$$= A\cos(\omega t) + \frac{F_0}{k}$$

Effectively driving force sets oscillations around a new zero-point $x \geq 0$ (if $F_0 > 0$)

For lower ω (higher m / lower k) zero-point shift larger



Unforced
↳ oscillations symmetric around origin
even/odd peaks identical in x^2!

Forced (high ω) Forced (low ω)
↳ odd peaks higher than even peaks

Peaks at $t = \frac{n\pi}{\omega}$, identical height in unforced case, odd peaks higher in forced case, especially for lower frequencies
Even peaks correspond to negative positions (against driving force)

So $\ddot{\Theta}_0 + k^2 c_s^2 \Theta_0 = F$ explains

oscillations

larger even/odd disparity as Ω_b is raised, since this lowers c_s^2 ~~(raises~~ (lowers ω)

Peaks at $\frac{n\pi}{\omega}$ shifted to larger k as $\Omega_b \uparrow$ $c_s^2 \downarrow$ and spacing between peaks gets larger

(think: $\frac{n\pi}{\omega} \propto \frac{1}{\omega} \propto \frac{1}{k^2 c_s^2}$ $\longrightarrow$ $c_s^2 \downarrow$ $k \uparrow$)

Another way to understand this:

- first peak has been growing since entered horizon

$\Omega_b \uparrow \Rightarrow c_s^2 \downarrow \Rightarrow P \downarrow \Rightarrow$ growth easier (peak is higher!)
more overdense

- second peak corresponds to underdensity

$\Omega_b \uparrow \Rightarrow c_s^2 \downarrow \Rightarrow P \downarrow \Rightarrow$ harder to escape well (peak is lower)
less underdense

Increasing Ω_b also damps the oscillations!

Photons have finite mean free path

$$\lambda_{MFP} \propto (n_e \sigma_T)^{-1}$$

scatters in Hubble time $\quad \dfrac{n_e \sigma_T}{H} = \dfrac{\lambda_H}{\lambda_{MFP}}$

cosmological photon over Hubble time moves

$$\lambda_D \sim \lambda_{MFP} \sqrt{N} = \lambda_{MFP} \sqrt{n_e \sigma_T H^{-1}} = \frac{1}{n_e \sigma_T} \sqrt{\frac{n_e \sigma_T}{H}} = \frac{1}{\sqrt{n_e \sigma_T H}}$$

Perturbations on scales $\lesssim \lambda_D$ are washed out
(high k modes damped)

$$\lambda_D \propto \frac{1}{\sqrt{n_e}} \propto \frac{1}{\sqrt{\Omega_b}} \qquad \Rightarrow k_D \propto \sqrt{\Omega_b}$$

when Universe ionized

So models with more baryons $\Omega_b \uparrow$ have $k_D \uparrow$ (damping occurs on smaller scales)

$k^{3/2} |\Theta_0 + \Psi| (\eta_*)$ gives perturbations at recombination
but we observe them today!

Photons from hot/cold spots typically separated by comoving distance k^{-1} $\rightarrow$ angular separation $\theta \simeq \frac{k^{-1}}{\eta_0-\eta_*} = \frac{1}{k(\eta_0-\eta_*)}$

In multipole moments

note: $\eta_0 \simeq 14$ Gpc *

$$\theta \sim \frac{1}{\ell}$$

$\eta_* \ll \eta_0$

$$\rightarrow \frac{1}{\ell} \sim \frac{1}{k(\eta_0-\eta_*)} \sim \frac{1}{k\eta_0} \Rightarrow \ell \simeq k\eta_0$$

$\cancel{\ell \simeq 1.4\times10^4\left(\frac{k}{\mathrm{Mpc}}\right)}$

$\cancel{\ell \simeq 1.4\times10^4 \frac{k}{\mathrm{Mpc}}}$

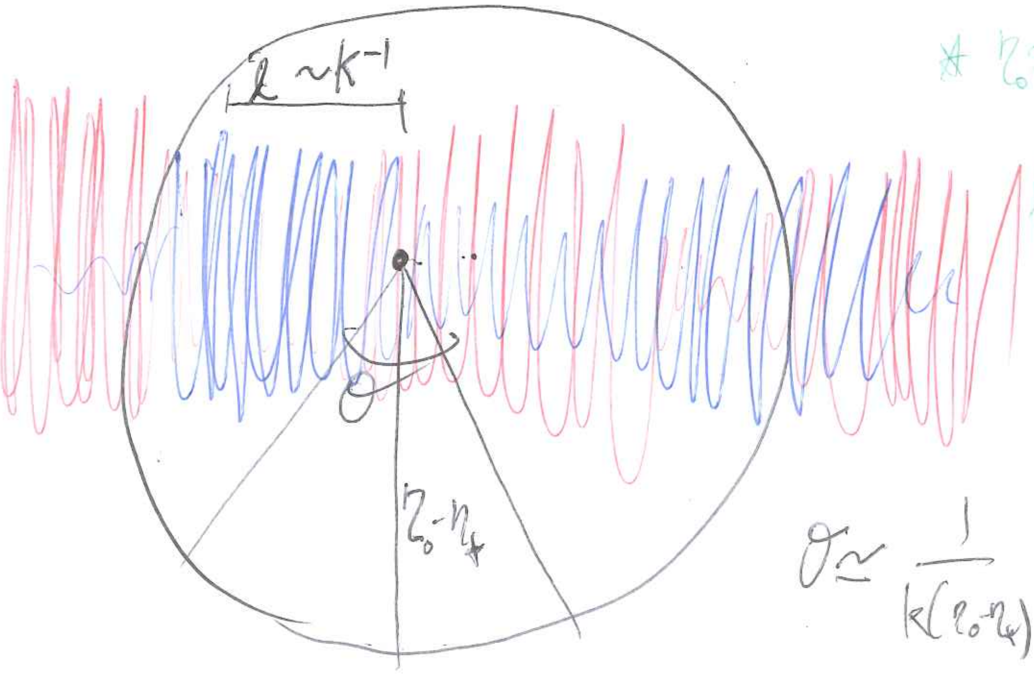
So inhomogeneities on scales k are projected roughly into anisotropies on angular scales $k\eta_0$

Modulo a few things which happen to photons on their journey from last-scattering to us

$$\left.\begin{cases} \phi \neq \text{const} \\ \text{Universe not flat?} \\ \text{lensing} \end{cases}\right] \text{Secondary anisotropies}$$

(~10% effect to anisotropies)

Pictorially, perturbation with wavenumber k



* $\eta_0 \simeq 14\,\mathrm{Gpc} = 1.4\times10^4\,\mathrm{Mpc}$

$\ell \simeq 1.4\times10^4\left(\frac{k}{\mathrm{Mpc}^{-1}}\right)$

$$\theta \simeq \frac{1}{k(\eta_0-\eta_*)}$$

Large-scale anisotropies

Recall equation for photon perturbations

$$\begin{cases} \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \\ \dot{\Theta}_1 - \frac{k}{3}\Theta_0 = -\frac{k}{3}\Phi \end{cases}$$

With initial conditions $\Phi(k,\eta_i) = -\Psi(k,\eta_i) = 2\Theta_0(k,\eta_i)$

$$\Theta_1(k,\eta_i) = -\frac{k\Phi(k,\eta_i)}{6aH}$$

Let's focus on the large-scale, super-horizon limit of the first equation

$k\eta \ll 1$ $\left(\dot{\Theta}_0 \sim \frac{\Theta_0}{\eta}\right)$ so $\frac{\dot{\Theta}_0}{k\Theta_1} \sim \frac{1}{k\eta} \gg 1$)

$$\dot{\Theta}_0 + \cancel{k\Theta_1} = -\dot{\Phi} \Rightarrow \dot{\Theta}_0 = -\dot{\Phi} \Rightarrow \Theta_0 = -\Phi + \text{const}$$

$\Theta_0 = -\Phi + \text{const}$ but we know $\Theta_0(k,\eta_i) = \frac{\Phi(k,\eta_i)}{2}$

$$\Rightarrow \text{const} = \frac{3\Phi_p}{2} \quad \text{so} \quad \Theta_0(k,\eta_i) = -\Phi_p + \frac{3\Phi_p}{2} = \frac{\Phi_p}{2} = \frac{\Phi(k,\eta_i)}{2}$$

$$\Theta_0 = -\Phi + \frac{3\Phi_p}{2}$$

We already got an expression for the large-scale evolution of Φ earlier (for modes that cross the horizon during MD and therefore are still super-horizon during RD)

$$\Phi = \frac{\Phi_p}{10}\frac{1}{y^3}\left[16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\right] \qquad y \equiv \frac{a}{a_{eq}}$$

$y \ll 1$: Φ_p $\qquad$ $y \gg 1$: $\frac{9\Phi_p}{10}$

Recombination takes place sufficiently after equality that we can use the $y \gg 1$ limit $\Phi \to \frac{9\Phi_p}{10}$ (at $\eta = \eta_*$)

Therefore $\Phi(k,\eta_*) = \frac{9}{10}\Phi_p(k) \to \Phi_p(k) = \frac{10}{9}\Phi(k,\eta_*)$

$$\Theta_0(k,\eta_*) = -\Phi(k,\eta_*) + \frac{3\Phi_p(k)}{2} = -\Phi(k,\eta_*) + \frac{\cancel{3}}{2}\frac{\cancel{10}^{5}}{\cancel{9}_3}\Phi(k,\eta_*) =$$

$$= -\Phi(k,\eta_*) + \frac{5}{3}\Phi(k,\eta_*) = \frac{2}{3}\Phi(k,\eta_*)$$

$$\Rightarrow \Theta_0(k,\eta_*) = \frac{2}{3}\Phi(k,\eta_*)$$

We argued that the observed anisotropy is $\sim \Theta_0 + \Psi$ to account for photon relative red/blueshift if $\Psi < 0$/$\Psi > 0$ since they have to travel out of under/over-densities at recombination

$$\Theta_0 + \Psi \sim \Theta_0 - \Phi \quad (\text{as } \Psi \sim -\Phi)$$

$$(\Theta_0 + \Psi)(k,\eta_*) \approx (\Theta_0 - \Phi)(k,\eta_*) = \frac{2}{3}\Phi(k,\eta_*) - \Phi(k,\eta_*) =$$

$$= -\frac{1}{3}\Phi(k,\eta_*) \approx \frac{1}{3}\Psi(k,\eta_*)$$

So ~~the~~ observed anisotropy $\sim \frac{1}{3}\Psi(k,\eta_*)$

Can also think in terms of density field. Recall

$\dot{\delta} + ikv = -3\dot{\Phi}$ with initial conditions $\delta(k,\eta_i) = \frac{3}{2}\Phi_p$

large-scale limit

$$\dot{\delta} = -3\dot{\Phi} \Rightarrow \cancel{\delta = -3\Phi + \text{const}}$$

$$\delta(k,\eta_*) = -3\Phi(k,\eta_*) + \text{const}$$

$$\cancel{\delta(k,\eta_*) = -3\Phi(}$$

$$\cancel{\delta(\eta_*) = -3\Phi(\eta_*) + \text{const} \xrightarrow{\eta_*}}$$

$$\delta(\eta) = -3\Phi(\eta) + \text{const} \xrightarrow{\eta \to 0} \delta = -3\Phi_p + \text{const} = \frac{3\Phi_p}{2}$$

$$\rightarrow \text{const} = \frac{3\Phi_p}{2} + 3\Phi_p = \frac{9\Phi_p}{2}$$

$$\delta(\eta_*) = -3\Phi(\eta_*) + \underbrace{\frac{9\Phi_p}{2}}_{\frac{9}{2}\Phi_p = \frac{3}{2}\Phi_p - (-3\Phi_p)} = \frac{3}{2}\Phi_p - 3\left[\Phi(\eta_*) - \Phi_p\right]$$

$$\cancel{\delta(\eta_*) = \frac{3}{2}\Phi_p - 3\left[\Phi(\eta_*) - \Phi_p\right] =}$$

$$\cancel{\Phi_p = \frac{10}{9}\Phi(\eta_*)} \qquad \cancel{\Phi(\eta_*) = \frac{9}{10}\Phi_p}$$

$$\delta(\eta_*) = -3\Phi(\eta_*) + \frac{9\Phi_p}{2} = -3\Phi(\eta_*) + \frac{\cancel{9}}{\cancel{2}}\frac{\cancel{10}^{\,5}}{\cancel{9}}\Phi(\eta_*) =$$

$$\Phi(\eta_*) = \frac{9}{10}\Phi_p \Rightarrow \Phi_p = \frac{10}{9}\Phi(\eta_*)$$

$$= -3\Phi(\eta_*) + 5\Phi(\eta_*) = 2\Phi(\eta_*)$$

$$\Rightarrow \delta(\eta_*) = 2\Phi(\eta_*) \qquad \Phi(\eta_*) = \frac{1}{2}\delta(\eta_*)$$

~~So primor~~ So observed anisotropy is

$$(\Theta_0 + \Psi)(k,\eta_*) = -\frac{1}{3}\Phi(k,\eta_*) = -\frac{1}{3}\frac{1}{2}\delta(\eta_*) = -\frac{1}{6}\delta(\eta_*)$$

Summarizing:

$$\Theta_0(k,\eta_*) = \frac{2\Phi(k,\eta_*)}{3} \xrightarrow{\text{observed anisotropy}} \boxed{(\Theta_0+\Psi)(k,\eta_*) = \begin{cases} -\frac{1}{3}\Phi(k,\eta_*) \\ \frac{1}{3}\Psi(k,\eta_*) \\ -\frac{1}{6}\delta(k,\eta_*) \end{cases}}$$

Consider

$$(\Theta_0+\Psi)(k,\eta_*) = -\frac{1}{6}\delta(\eta_*)$$

$$\Big\downarrow \text{ Fourier transform...}$$

$$\tilde{\Theta}_0+\tilde{\Psi} < 0 \quad \text{for} \quad \delta>0 \ (\Psi<0)$$

$\downarrow$

observed anisotropy of an overdense region is negative!
(cold spot)

So for large-scale perturbations cold spots correspond to overdense regions at recombination. These were hotter at recombination ($\Theta_0>0$ when $\Psi<0$), however to travel to us they need to climb out of potential wells, and they lose ~~more~~ energy in a way which more than compensates for their originally being hotter, i.e. $\Theta_0+\Psi<0$ when $\Psi<0$

cold spots $\to$ overdensities at η_* hot spots $\to$ underdensities at η_*

Factor $\frac{1}{6}$ is interesting

$$\cancel{(\Theta_0+\Psi)}$$

$$\frac{\delta T}{T} \sim |\Theta_0+\Psi| \sim \frac{1}{6}\delta \sim \frac{1}{6}\frac{\delta\rho}{\rho}$$

So $\sim 10^{-5}$ anisotropies correspond to $\sim 6\times10^{-5}$ overdensities, so we can ask the question of whether the observed anisotropies are consistent with the overdensities needed to form structure (yes! In NCDM with inflation)

Acoustic oscillations

For $\eta \ll \eta_*$, electrons ionized, $\lambda_{MFP} \ll H^{-1}$, γ-b tightly coupled

Tightly coupled limit: $\tau \gg 1$ $\left[\tau \equiv \int_\eta^{\eta_0} d\eta' a n_e \sigma_T\right]$

In $\tau \gg 1$ limit only non-negligible moments are Θ_0 and Θ_1

(so γ are a fluid just described by ρ and $\vec{v}$)

Consider Boltzmann equation for γ

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_e - \frac{1}{2}P_2(\mu)\Pi\right]$$

Turn it into infinite set of coupled equations for Θ_ℓ

Recall

$$\Theta_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)$$

Neglect polarization

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_b\right]$$

Consider $\ell \geq 2$ moments, multiply by $P_\ell(\mu)$, integrate over μ

$$\int\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu) = (-i)^\ell\Theta_\ell \qquad \int d\mu\, P_\ell(\mu)\Theta(\mu) = 2(-i)^\ell\Theta_\ell$$

$$\boxed{\frac{1}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\dot{\Theta}(\mu)} + \boxed{\frac{1}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\,ik\mu\Theta(\mu)} = \boxed{\frac{\dot{\tau}}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\Theta(\mu)} \qquad \boxed{\ell \geq 2}$$

$$\boxed{-\frac{1}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\dot{\Phi} - \frac{ik}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\Psi - \frac{\dot{\tau}}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\Theta_0 - \frac{\dot{\tau}v_b}{(-i)^\ell}\int\frac{d\mu}{2}P_\ell(\mu)\mu}$$

$\boxed{}$: terms all of the form

$$\propto \int d\mu\, P_\ell(\mu)\,\mu \qquad \text{or} \qquad \int d\mu\, P_\ell(\mu)$$

$$\propto \int d\mu\, P_{\ell\geq 2}(\mu) P_1(\mu) \qquad\qquad \propto \int d\mu\, P_{\ell\geq 2}(\mu) P_0(\mu)$$

$$\propto \delta_{\ell\geq 2,1} = 0 \qquad\qquad \propto \delta_{\ell\geq 2,0} = 0$$

So all $\boxed{}$ terms $= 0$

$$\frac{1}{(-i)^\ell}\int \frac{d\mu}{2} P_\ell(\mu)\dot\Theta(\mu) = \frac{1}{(-i)^\ell}\int \frac{d\mu}{2} P_\ell(\mu)\frac{\partial\Theta}{\partial\eta}(\mu) = \frac{\partial}{\partial\eta}\left[\frac{1}{(-i)^\ell}\int\frac{d\mu}{2} P_\ell(\mu)\Theta(\mu)\right] =$$

$$= \frac{\partial}{\partial\eta}\Theta_\ell = \dot\Theta_\ell$$

$$\frac{\dot\tau}{(-i)^\ell}\int\frac{d\mu}{2} P_\ell(\mu)\Theta(\mu) = \dot\tau\,\Theta_\ell$$

$$\frac{1}{(-i)^\ell}\int\frac{d\mu}{2} P_\ell(\mu)\, ik\mu\,\Theta(\mu) = \frac{k}{(-i)^{\ell+1}}\int\frac{d\mu}{2} P_\ell(\mu)\,\mu\,\Theta(\mu)$$

Use recurrence relation for Legendre polynomials

$$(\ell+1)P_{\ell+1}(\mu) = (2\ell+1)\mu P_\ell(\mu) - \ell P_{\ell-1}(\mu)$$

$$\Rightarrow \mu P_\ell(\mu)\Theta(\mu) = \frac{\ell+1}{2\ell+1}P_{\ell+1}(\mu) + \frac{\ell}{2\ell+1}P_{\ell-1}(\mu)$$

$$\frac{1}{(-i)^{\ell+1}} = \frac{1}{(-i)^{\ell-1}}\frac{1}{(-i)^2}$$

$$\Theta_{\ell-1} \qquad \boxed{(-i)^2}$$

$$\Rightarrow \frac{k}{(-i)^{\ell+1}}\int\frac{d\mu}{2}\mu P_\ell(\mu)\Theta(\mu) = \frac{k(\ell+1)}{(2\ell+1)}\underbrace{\frac{1}{(-i)^{\ell+1}}\int\frac{d\mu}{2}P_{\ell+1}(\mu)\Theta(\mu)}_{\Theta_{\ell+1}} + \frac{k\ell}{(2\ell+1)}\frac{1}{(-i)^{\ell+1}}\int\frac{d\mu}{2}P_{\ell-1}(\mu)\Theta(\mu) =$$

$$= \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1} + \frac{k\ell}{(2\ell+1)}\frac{1}{(-i)^2}\Theta_{\ell-1} = \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1} - \frac{k\ell}{2\ell+1}\Theta_{\ell-1}$$