Diffusion damping

Recall equation for photon distribution moments

$$\begin{cases} \dot{\Theta}_\ell - \frac{k\ell}{2\ell+1}\Theta_{\ell-1} + \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1} = \dot{\tau}\Theta_\ell \\ \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \\ \dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\Psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{i v_b}{3}\right] \end{cases}$$

Diffusion damping characterized by small but non-negligible Θ_2

We now care only about very small scales, where Φ, Ψ are negligible as they are damped after horizon crossing $\left[\Phi \sim \frac{\sin/\cos(k\eta)}{(k\eta)^3}\right]$

$\Downarrow$

$$\begin{cases} \dot{\Theta}_0 + k\Theta_1 = 0 \\ \dot{\Theta}_1 + \boxed{k\left(\frac{2}{3}\Theta_2\right.}\left.- \frac{1}{3}\Theta_0\right) = \dot{\tau}\left[\Theta_1 - \frac{i v_b}{3}\right] \\ \boxed{\dot{\Theta}_2 - \frac{2k}{5}\Theta_1 = \frac{9}{10}\dot{\tau}\Theta_2} \end{cases}$$

$\boxed{} = \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1}$ with $\ell = 1$

$\rightarrow$ by doing $\int \times \frac{d\mu}{2} P_2(\mu)$

Θ_ℓ, $\ell \geq 3$ neglected as higher moments are suppressed by $\frac{1}{\dot{\tau}}$

We also need equation for v_b

Start from earlier

neglect Ψ!

$$v_b \simeq -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a}v_b + ik\Psi\right] \simeq -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a}v_b\right]$$

$$\Rightarrow 3i\Theta_1 + v_b \simeq \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a}v_b\right]$$

equation for v_b

Use WKB-like approximation given that damping is a high-frequency phenomenon (many damping in $\sim \frac{1}{H}$)

$$v_b \propto e^{i\int \omega \, d\eta}$$

and similarly for $\Theta_0, \Theta_1, \Theta_2$

In the tightly coupled limit $\omega \simeq k c_s$ real. Damping corresponds to $\mathrm{Im}(\omega)$

$$v_b \propto e^{i\int d\eta\, \omega(\eta)} \longrightarrow \dot{v}_b = i\omega e^{i\int d\eta\, \omega(\eta)} = i\omega v_b \gg \frac{\dot{a}}{a} v_b$$

Damping frequency $\gg$ "Expansion frequency" as $\frac{\dot{a}}{a} \propto \frac{1}{\eta}$, $\omega \propto k$

$$3i\Theta_1 + v_b = \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a} v_b\right] \simeq \frac{R}{\dot{\tau}} \dot{v}_b = \frac{R}{\dot{\tau}} i\omega v_b$$

$$\Rightarrow v_b + 3i\Theta_1 \simeq \frac{R}{\dot{\tau}} i\omega v_b \Longrightarrow v_b\left[1 - \frac{i\omega R}{\dot{\tau}}\right] \simeq -3i\Theta_1$$

$$\Rightarrow v_b \simeq -3i\Theta_1 \left[1 - \frac{i\omega R}{\dot{\tau}}\right]^{-1} \simeq -3i\Theta_1\left[1 + \frac{i\omega R}{\dot{\tau}} - \left(\frac{\omega R}{\dot{\tau}}\right)^2\right]$$

$(1-x)^{-1} \simeq 1 + x + x^2 + \mathcal{O}(x^3)$

Expand to $\frac{1}{\dot{\tau}^2}$ as $(v_b + 3i\Theta_1)$ multiplied by $\dot{\tau}$ in $\dot{\Theta}_1$ equation

Similarly reduce Θ_2 equation

$$\cancel{\dot{\Theta}_2} - \frac{2k}{5}\Theta_1 = \frac{9}{10}\dot{\tau}\Theta_2 \xrightarrow[\dot{\tau} \gg \frac{1}{\eta}]{\dot{\Theta}_2 \ll \frac{9}{10}\dot{\tau}\Theta_2} \Theta_2 = -\frac{2k}{5}\frac{10}{9\dot{\tau}}\Theta_1 = -\frac{4k}{9}\Theta_1$$

So $\Theta_2 = \frac{4k}{9\dot{\tau}}\Theta_1$ confirming that higher moments are indeed supressed by $\frac{k}{\dot{\tau}} \ll 1 \Longrightarrow \dot{\tau} \gg k$

~~Zero~~ Same with equation for θ_0

$$\dot{\theta}_0 + k\theta_1 = 0 \qquad \theta_0 \propto e^{i\int d\eta\, \omega(\eta)} \Rightarrow \dot{\theta}_0 = i\omega\theta_0$$

$$\Rightarrow i\omega\theta_0 + k\theta_1 = 0 \Rightarrow i\omega\theta_0 = -k\theta_1$$

Overall we get

$$\begin{cases} i\omega\theta_0 \simeq -k\theta_1 & \longrightarrow -\frac{1}{3}\theta_0 \simeq +\frac{k}{3i\omega}\theta_1 \\ \theta_2 \simeq -\frac{4k}{9\dot{\tau}}\theta_1 & \longrightarrow \frac{2}{3}\theta_2 \simeq -\frac{8k}{27\dot{\tau}}\theta_1 \\ v_b \simeq -3i\theta_1\left[1 + \frac{i\omega R}{\dot{\tau}} - \left(\frac{\omega R}{\dot{\tau}}\right)^2\right] & \longrightarrow -\frac{iv_b}{3} \simeq -\theta_1\left[1 + \frac{i\omega R}{\dot{\tau}} - \left(\frac{\omega R}{\dot{\tau}}\right)^2\right] \end{cases}$$

Put all these inside equation for θ_1

$$\dot{\theta}_1 + k\left(\frac{2}{3}\theta_2 - \frac{1}{3}\theta_0\right) = \dot{\tau}\left(\theta_1 - \frac{iv_b}{3}\right) \qquad \theta_1 \propto e^{i\int d\eta\, \omega(\eta)} \Rightarrow \dot{\theta}_1 = i\omega\theta_1$$

$$\Rightarrow i\omega\cancel{\theta_1} - \frac{8k^2}{27\dot{\tau}}\cancel{\theta_1} + \frac{k^2}{3i\omega}\cancel{\theta_1} = \dot{\tau}\left(1 - \left[1 + \frac{i\omega R}{\dot{\tau}} - \left(\frac{\omega R}{\dot{\tau}}\right)^2\right]\right)\cancel{\theta_1}$$

$$\Rightarrow i\omega - \frac{8k^2}{27\dot{\tau}} + \frac{k^2}{3i\omega} = \dot{\tau}\left(1 - \left[1 + \frac{i\omega R}{\dot{\tau}} - \left(\frac{\omega R}{\dot{\tau}}\right)^2\right]\right)$$

$$\Rightarrow i\omega - \frac{8k^2}{27\dot{\tau}} + \frac{k^2}{3i\omega} = \cancel{\dot{\tau}} - \cancel{\dot{\tau}} - i\omega R + \frac{(\omega R)^2}{\dot{\tau}}$$

$\times -i\omega$

$$\Rightarrow \omega^2 + \frac{8ik^2}{27\dot{\tau}}\omega - \frac{k^2}{3} = -\omega^2 R - i\frac{(\omega R)^2}{\dot{\tau}}\omega$$

$$\Rightarrow \boxed{\omega^2(1+R) - \frac{k^2}{3} + \frac{i\omega}{\dot{\tau}}\left[\omega^2 R^2 + \frac{8k^2}{27}\right] = 0}$$

Dispersion relation

Consider no damping, ~~also~~ neglect $\dot{\tau}$ term

$$\Rightarrow \omega^2(1+R) = \frac{k^2}{3} \longrightarrow \omega^2 = \frac{k^2}{3(1+R)}$$

standard dispersion relation

$\omega^2 = c^2k^2$ with $c = \sqrt{\frac{1}{3(1+R)}}$

With damping there is an imaginary part

$\omega = \omega_0 + \delta\omega \qquad \omega_0^2 = c_s^2 k^2 \qquad \omega_0 = c_s k$

$\omega = c_s k + \delta\omega \qquad \delta\omega$ first order in $\frac{1}{\dot{\tau}}$

↳ re-insert into dispersion relation, look at 1st order terms

$$\underbrace{\cancel{c_s^2 k^2 (1+R)}}_{=\frac{k^2}{3}\text{ since } c_s^2 = \frac{1}{3(1+R)}} + 2c_s k(1+R)\delta\omega - \cancel{\frac{k^2}{3}} + \frac{ic_s k}{\dot{\tau}}\left[c_s^2 k^2 R^2 + \frac{8k^2}{27}\right] = 0$$

$$\Rightarrow 2\cancel{c_s}k(1+R)\delta\omega + \frac{i\cancel{c_s}k}{\dot{\tau}}\left[c_s^2 k^2 R^2 + \frac{8k^2}{27}\right] = 0$$

$$\Rightarrow \boxed{\delta\omega \simeq -\frac{ik^2}{2(1+R)\dot{\tau}}\left[c_s^2 R^2 + \frac{8}{27}\right]}$$

Recall $\Theta_0, \Theta_1 \propto \exp\left[i\int d\eta\, \omega(\eta)\right] = \exp\left\{i\int d\eta\left[\omega(\eta) + \delta\omega(\eta)\right]\right\}$

So $\omega \simeq \omega_0 + \delta\omega = c_s k + \delta\omega \rightarrow i\omega \simeq ic_s(\eta)k + i\delta\omega(\eta)$

$$\Theta_0, \Theta_1 \propto \exp\left\{ik\int d\eta\, c_s(\eta)\right\}\exp\left\{-\frac{k^2}{k_D^2}\right\}$$

where we defined

$$\frac{1}{k_D^2(\eta)} \equiv \int_0^\eta \frac{d\eta'}{6(1+R)n_e\sigma_T a(\eta')}\left[\frac{R^2}{(1+R)} + \frac{8}{9}\right]$$

Where does this come from?

$$\Theta_0, \Theta_1 \propto e^{i\int d\eta\, \delta\omega(\eta)}$$

$$\dot\tau = -a n_e \sigma_T$$
$$c_s^2 = \frac{1}{3(1+R)}$$

$$i\,\delta\omega(\eta) = \frac{K^2}{2(1+R)\dot\tau}\left[c_s^2 R^2 + \frac{8}{27}\right] =$$

$$= -\frac{K^2}{2(1+R)a n_e \sigma_T}\left[\frac{R^2}{3(1+R)} + \frac{8}{27}\right] = -\frac{K^2}{6(1+R)n_e\sigma_T a}\left[\frac{R^2}{(1+R)} + \frac{8}{9}\right]$$

$$\int d\eta\, i\,\delta\omega(\eta) = -K^2\int_0^{\eta} d\eta' \frac{1}{6(1+R)n_e\sigma_T a(\eta')}\left[\frac{R^2}{(1+R)} + \frac{8}{9}\right] \equiv -\frac{K^2}{K_D^2}$$

So $\dfrac{1}{K_D} \sim \sqrt{\dfrac{\eta}{n_e\sigma_T a}}$ as we had guessed earlier

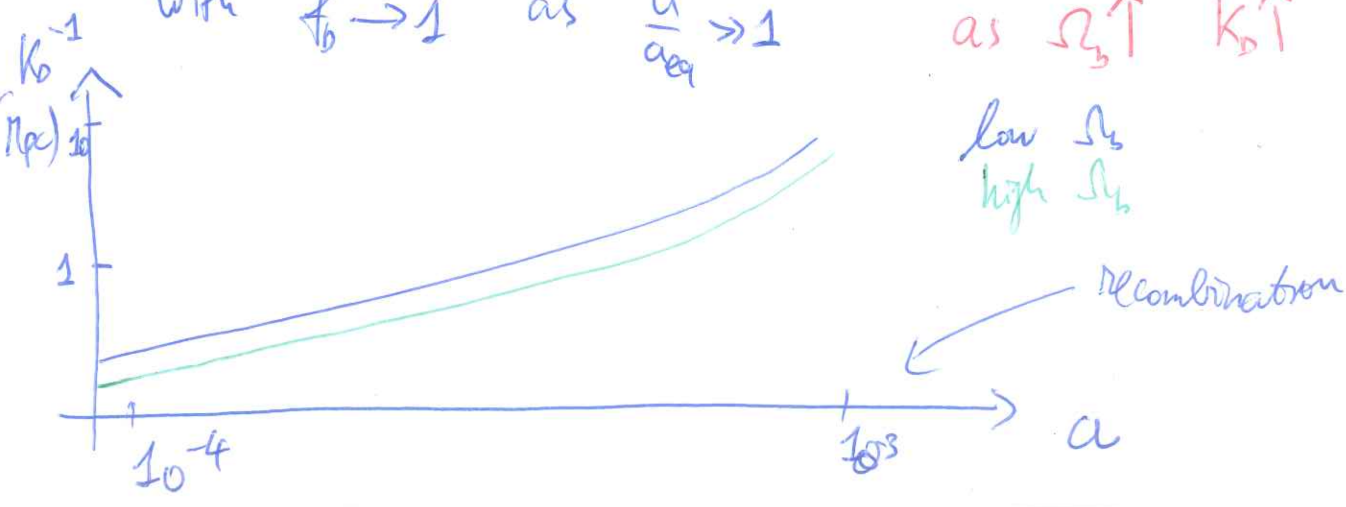
To understand K_D, work in prerecombination limit, all electrons ionized except in Helium.

$$n_e\sigma_T a \simeq 2.3\times10^{-5}\,\mathrm{Mpc}^{-1}\,\Omega_b h^2 a^{-2}\left(1-\frac{Y_p}{2}\right)$$

$$\rightarrow K_D^{-2} \simeq 3.1\times10^{6}\,\mathrm{Mpc}^2\, a^{5/2} f_D\!\left(\frac{a}{a_{eq}}\right)(\Omega_b h^2)^{-1}\left(1-\frac{Y_p}{2}\right)^{-1}$$

with $f_D \to 1$ as $\dfrac{a}{a_{eq}} \gg 1$

as $\Omega_b \uparrow$ $K_D \uparrow$ $K_D^{-1} \downarrow$



Without recombination $k_D \propto \sqrt{\Omega_b}$

Damping of anisotropies due to photon diffusion <u>not</u> simply due to finite thickness of last-scattering surface, because even before recombination $\dot{\tau} < \infty$. ~~Even if recombination instantaneous~~

<u>Inhomogeneities to anisotropies</u>

We want to express $\Theta_\ell(k, \eta_0)$ as a function of $\Theta_0(k, \eta_*)$ and $\Theta_1(k, \eta_*)$, then relate these to the CMB C_ℓs.

First we need to solve for $\Theta_\ell(\eta_0)$ in terms of $\Theta_{0,1}(k, \eta_*)$

Consider photon Boltzmann equation

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

$\cancel{\text{add } \dot{\Theta}}$ $-\dot{\tau}\Theta \Rightarrow$

$$\boxed{\dot{\Theta} + ik\mu\Theta - \dot{\tau}\Theta}^{*} = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

$$* \quad \dot{\Theta} + ik\mu\Theta - \dot{\tau}\Theta = \dot{\Theta} + (ik\mu - \dot{\tau})\Theta = e^{-ik\mu\eta + \tau}\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta - \tau}\right]$$

check $e^{-ik\mu\eta+\tau}\dfrac{d}{d\eta}\left[\Theta e^{ik\mu\eta - \tau}\right] = e^{-ik\mu\eta+\tau}\left[\dot{\Theta}e^{ik\mu\eta-\tau} + (ik\mu - \dot{\tau})\Theta e^{ik\mu\eta-\tau}\right] =$

$$= \dot{\Theta} + (ik\mu - \dot{\tau})\Theta$$

So we can rewrite photon equation as

$$e^{-ik\mu\eta+\tau}\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right] = \tilde{S}$$

where source function defined as

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

$$\Rightarrow \frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right] = \tilde{S} e^{ik\mu\eta-\tau(\eta)}$$

$$\cancel{\Rightarrow \Theta e^{ik\mu\eta-\tau} = \int_{\eta_i}^{\eta_0} d\eta\, \tilde{S}(\eta) e^{ik}}$$

~~Integrate to get $\Theta(\eta_0)$ [from η_i to η_0]~~

$$\cancel{\Theta(\eta_0) = \Theta(\eta_i) e^{ik\mu(\eta_i-\eta_0)} e^{-\tau(\eta_i-\eta_0)} + \int_{\eta_i}^{\eta} d\eta\, \tilde{S}(\eta) e^{ik\mu(\eta-\eta_0)-\tau(\eta-\eta_0)}}$$

~~Integrate to get $\Theta(\eta_0)$ from η_i to η~~

$$\cancel{\Theta e^{ik\mu\eta-\tau(\eta)} = \int_{\eta_i}^{\eta} d\eta\, \tilde{S}(\eta) e^{ik\mu}}$$

Integrate to get $\Theta(\eta_0)$

$$\Theta(\eta_0) = \Theta(\eta_i) e^{ik\mu(\eta_i-\eta_0)} e^{-\tau(\eta_i)} + \int_{\eta_i}^{\eta_0} d\eta\, \tilde{S}(\eta) e^{ik\mu(\eta-\eta_0)-\tau(\eta)}$$

[using $\tau(\eta_0)=0$ since $\tau \equiv \int_{\eta}^{\eta_0} d\eta'\, n_e \sigma_T a(\eta)$]

If $\eta_i \ll \eta_0$, τ very large at η_i so anything multiplied by $e^{-\tau(\eta_i)}$ can be neglected. Physically: Compton scattering erases initial anisotropies, so we may actually set $\eta_i \sim 0$

$$\Rightarrow \boxed{\Theta(k,\mu,\eta_0) \simeq \int_0^{\eta_0} d\eta\, \tilde{S}(k,\mu,\eta) e^{ik\mu(\eta-\eta_0)-\tau(\eta)}}$$

$\tilde{S}$ (which also depends on μ) implicitly hides all the complication

If $\tilde{S}$ did not depend on μ this could immediately become an equation for each of the Θ_ℓ s. Let's pretend this is the case... (forget for the moment μ-dependence of $\tilde{S}$)

$$\underbrace{\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\,\Theta(k,\mu,\eta_0)}_{=(-i)^\ell\Theta_\ell} \simeq \int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\int_0^{\eta_0} d\eta\,\tilde{S}(k,\eta)e^{ik\mu(\eta-\eta_0)-\tau(\eta)}$$

$$=\int_0^{\eta_0}\underbrace{\left[\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)e^{ik\mu(\eta-\eta_0)}\right]}_{\frac{1}{(-i)^\ell}j_\ell[k(\eta-\eta_0)]}\tilde{S}(k,\eta)e^{-\tau(\eta)}$$

j_ℓ spherical Bessel function

$$\Rightarrow (-i)^\ell\Theta_\ell \simeq \frac{1}{(-i)^\ell}\int_0^{\eta_0} d\eta\,\tilde{S}(k,\eta)e^{-\tau(\eta)}j_\ell[k(\eta-\eta_0)]$$

$$\Rightarrow \boxed{\Theta_\ell(k,\eta_0) \simeq (-1)^\ell\int_0^{\eta_0} d\eta\,\tilde{S}(k,\eta)e^{-\tau(\eta)}j_\ell[k(\eta-\eta_0)]}$$

But $\tilde{S}$ actually depends on μ. Note however that we can "replace" $\mu \longrightarrow \frac{1}{ik}\frac{d}{d\eta}$ in $\tilde{S}$ since it multiplies $e^{ik\mu(\eta-\eta_0)}$

$$\tilde{S} = \dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

e.g. consider $-ik\mu$

Explicitly

$$-ik\int_0^{\eta_0} d\eta\,\mu\Psi e^{ik\mu(\eta-\eta_0)}e^{-\tau(\eta)} = -\int_0^{\eta_0} d\eta\,\Psi e^{-\tau(\eta)}\underbrace{\boxed{\frac{d}{d\eta}\left[e^{ik\mu(\eta-\eta_0)}\right]}}_{=ik\mu e^{ik\mu(\eta-\eta_0)}} =$$

$$= -\Psi e^{-\tau(\eta)}e^{ik\mu(\eta-\eta_0)}\Big|_0^{\eta_0} + \int_0^{\eta_0} d\eta\,e^{ik\mu(\eta-\eta_0)}\frac{d}{d\eta}\left[\Psi e^{-\tau(\eta)}\right] =$$

$$= \cancel{+\Psi(0) e^{\tau(0)} e^{ik\mu\ell}} \quad \tau(0) \text{ very large}$$

$$= \underbrace{\Psi(\eta_0) e^{-\tau(\eta_0)}} - \cancel{\Psi(0) e^{-\tau(0)} e^{-ik\mu\eta_0}} + \int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)} \frac{d}{d\eta}\left[\Psi e^{-\tau(\eta)}\right]$$

$\tau(0)$ very large

Just "redefinition" of the monopole which cannot be detected no angular dependence

$$\text{"="} \int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)} \frac{d}{d\eta}\left[\Psi e^{-\tau(\eta)}\right]$$

for our purposes

Since

$$-ik\int_0^{\eta_0} d\eta\, \mu\, \Psi e^{ik\mu(\eta-\eta_0)} e^{-\tau(\eta)} \Rightarrow \int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)} \frac{d}{d\eta}\left[\Psi e^{-\tau(\eta)}\right]$$

This is basically a substitution

$$-ik\mu \longrightarrow \frac{d}{d\eta} \qquad \text{i.e.} \qquad \mu \rightarrow \frac{1}{ik}\frac{d}{d\eta}$$

Note: derivative does **NOT** act on $e^{ik\mu(\eta-\eta_0)}$

So we can take

$$\Theta_\ell(k,\eta_0) = (-1)^\ell \int_0^{\eta_0} d\eta\, \tilde{S}(k,\eta) e^{-\tau(\eta)} J_\ell\left[k(\eta-\eta_0)\right]$$

$(-1)^\ell J_\ell(x) = J_\ell(-x)$

$$= \int_0^{\eta} d\eta\, \tilde{S}(k,\eta) e^{-\tau(\eta)} J_\ell\left[k(\eta_0-\eta)\right]$$

and write it as

$$\Theta_\ell(k,\eta_0) = \int_0^{\eta_0} d\eta\, S(k,\eta) J_\ell\left[k(\eta_0-\eta)\right]$$

With a redefined source function

$$S(k,\eta) = \tilde{S}(k,\eta,\mu)e^{-\tau(\eta)}\Big|_{\mu\to\frac{1}{ik}\frac{d}{d\eta}\,,\ \mu^2\to-\frac{1}{k^2}\frac{d^2}{d\eta^2}}$$

and therefore

$$S(k,\eta) = e^{-\tau(\eta)}\left[-\dot{\Phi}-\dot{\tau}\left(\Theta_0+\frac{1}{4}\Pi\right)^{*}\right] \quad \Big\}\text{ no } \mu \text{ dependence}$$

$$+\frac{d}{d\eta}\left[e^{-\tau(\eta)}\left(\Psi-\frac{iv_b\dot{\tau}}{k}\right)\right] \quad \Big\}\propto\mu$$

$$-\frac{3}{4k^2}\frac{d^2}{d\eta^2}\left[e^{-\tau}\dot{\tau}\Pi\right]^{*} \quad \Big\}\propto\mu^2$$

* $\tilde{S}$ contains $-\dot{\tau}\times-\frac{1}{2}P_2(\mu)\Pi = \frac{\dot{\tau}}{2}P_2(\mu)\Pi = \frac{\dot{\tau}}{2}\left(\frac{3\mu^2-1}{2}\right)\Pi =$

$$= \frac{3}{4}\mu^2\dot{\tau}\Pi-\frac{1}{4}\dot{\tau}\Pi = \dot{\tau}\Pi\left(\frac{3}{4}\mu^2-\frac{1}{4}\right) \Longrightarrow -\frac{3}{4k^2}\frac{d^2}{d\eta^2}\dot{\tau}\Pi-\frac{\dot{\tau}\Pi}{4}$$

Overall source function

$$\boxed{S(k,\eta) = e^{-\tau}\left[-\dot{\Phi}-\dot{\tau}\left(\Theta_0+\frac{\Pi}{4}\right)\right]+\frac{d}{d\eta}\left[e^{-\tau}\left(\Psi-\frac{iv_b\dot{\tau}}{k}\right)\right]-\frac{3}{4k^2}\frac{d^2}{d\eta^2}\left[e^{-\tau}\dot{\tau}\Pi\right]}$$

Define visibility function

$$g(\eta) \equiv -\dot{\tau}e^{-\tau}$$

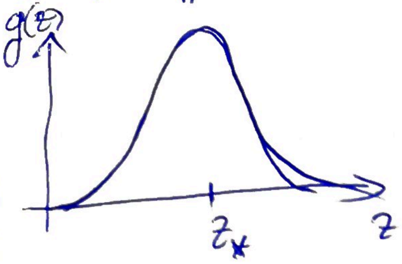
Properties: $\int_0^{\eta_0} d\eta\, g(\eta) = -\int_0^{\eta_0} d\eta\left(\frac{d\tau}{d\eta}\right)e^{-\tau} = -\tau e^{-\tau}\Big|_0^{\eta_0}+\int_0^{\eta_0} d\eta\, \tau\frac{d}{d\eta}\left(e^{-\tau}\right) =$

$$= \underbrace{\cancel{\tau(\eta_0)e^{-\tau(\eta_0)}}}_{\tau(\eta_0)=0} - \underbrace{\cancel{\tau(0)e^{-\tau(0)}}}_{\tau(0)\to\infty} - \int_0^{\eta_0} d\eta\,\tau e^{-\tau} = e^{-\tau}(1+\tau)\Big|_0^{\eta_0} = e^{-\tau(\eta_0)}(1+\tau(\eta_0))-e^{-\tau(0)}(1+\tau(0))$$

$$\simeq e^{-0}(1+0) = 1\times1 = 1!$$

Since $\int d\eta\, g(\eta)=1$, $g(\eta)\equiv -\dot{\tau}e^{-\tau}$ is like a probability density: probability photon last-scattered at η!

$g(\eta)$ suppressed at $\xrightarrow[\eta\to 0]{}$ because $\tau(\eta)\xrightarrow[\eta\to 0]{}\infty$ $e^{-\tau(\eta)}\to 0$!!)



$\eta \gg \eta_*$ → ~~because~~

because $\tau(\eta)\to 0$ $\left[\tau\sim\int_{\eta}^{\eta_0} d\eta\, a n_e \sigma_T\right]$

Visibility function peaked at $z\sim z_*$ with longer tail at low redshift (it's not symmetrical around z_*)

let's express $S(k,\eta)$ in terms of $g(\eta)$, also dropping Π

$$S(k,\eta)=e^{-\tau}\left[-\dot{\Phi}-\dot{\tau}\left(\Theta_0+\cancel{\frac{\Pi}{4}}\right)\right]+\frac{d}{d\eta}\left[e^{-\tau}\left(\Psi-\frac{iv_b\dot{\tau}}{k}\right)\right]-\frac{3}{4k^2}\frac{d^2}{d\eta^2}\left[e^{-\tau}\cancel{\dot{\tau}\Pi}\right]$$

$$\simeq -e^{-\tau}\dot{\Phi}\underbrace{-\dot{\tau}e^{-\tau}}_{g(\eta)}\Theta_0\underbrace{-\dot{\tau}e^{-\tau}}_{g(\eta)}\Psi+e^{-\tau}\dot{\Psi}+\frac{d}{d\eta}\left[\underbrace{-\dot{\tau}e^{-\tau}}_{g(\eta)}\frac{iv_b}{k}\right]=$$

$$=g(\eta)\left[\Theta_0(k,\eta)+\Psi(k,\eta)\right]+\frac{d}{d\eta}\left[\frac{iv_b(k,\eta)g(\eta)}{k}\right]+e^{-\tau}\left[\dot{\Psi}(k,\eta)-\dot{\Phi}(k,\eta)\right]$$

Recall $\Theta_\ell \Leftrightarrow S(k,\eta)$ relation.

$$\Theta_\ell(k,\eta_0)=\int_0^{\eta_0}d\eta\, S(k,\eta)\, j_\ell[k(\eta_0-\eta)]=$$

by parts since $g(\eta_0)\to 0$, $g(0)\to 0$

$$=\int_0^{\eta_0}d\eta\, g(\eta)\left[\Theta_0(k,\eta)+\Psi(k,\eta)\right]j_\ell[k(\eta_0-\eta)] \quad (1) \quad -\int_0^{\eta_0}d\eta\, g(\eta)\frac{iv_b(k,\eta)}{k}\frac{d}{d\eta}j_\ell[k(\eta_0-\eta)] \quad (2)$$

$$+\int_0^{\eta_0}d\eta\, e^{-\tau}\left[\dot{\Psi}(k,\eta)-\dot{\Phi}(k,\eta)\right]j_\ell[k(\eta_0-\eta)] \quad (3)$$

3 terms: (1), (2), (3). Study each one