$\boxed{}$: terms all of the form

$$\propto \underbrace{\int d\mu\, P_\ell(\mu)\,\mu}_{\propto \int d\mu\, P_{\ell\geq 2}(\mu) P_1(\mu)} \qquad \text{or} \qquad \underbrace{\int d\mu\, P_\ell(\mu)}_{\propto \int d\mu\, P_{\ell\geq 2}(\mu) P_0(\mu)}$$

$$\propto \delta_{\ell\geq 2, 1} = 0 \qquad\qquad \propto \delta_{\ell\geq 2, 0} = 0$$

So all $\boxed{}$ terms $= 0$

$$\frac{1}{(-i)^\ell}\int \frac{d\mu}{2} P_\ell(\mu)\dot{\Theta}(\mu) = \frac{1}{(-i)^\ell}\int \frac{d\mu}{2} P_\ell(\mu)\frac{\partial \Theta}{\partial \eta}(\mu) = \frac{\partial}{\partial \eta}\left[\frac{1}{(-i)^\ell}\int \frac{d\mu}{2} P_\ell(\mu)\Theta(\mu)\right] =$$

$$= \frac{\partial}{\partial \eta}\Theta_\ell = \dot{\Theta}_\ell$$

$$\frac{\dot{\tau}}{(-i)^\ell}\int \frac{d\mu}{2} P_\ell(\mu)\Theta(\mu) = \dot{\tau}\Theta_\ell$$

$$\frac{1}{(-i)^\ell}\int \frac{d\mu}{2} P_\ell(\mu)\, i k \mu\, \Theta(\mu) = \frac{k}{(-i)^{\ell+1}}\int \frac{d\mu}{2} P_\ell(\mu)\,\mu\,\Theta(\mu)$$

Use recurrence relation for Legendre polynomials

$$(\ell+1)P_{\ell+1}(\mu) = (2\ell+1)\mu P_\ell(\mu) - \ell P_{\ell-1}(\mu)$$

$$\Rightarrow \mu P_\ell(\mu)\Theta(\mu) = \frac{\ell+1}{2\ell+1}P_{\ell+1}(\mu) + \frac{\ell}{2\ell+1}P_{\ell-1}(\mu)$$

$$\frac{1}{(-i)^{\ell+1}} = \frac{1}{(-i)^{\ell-1}}\frac{1}{(-i)^2}$$

$$\Rightarrow \frac{k}{(-i)^{\ell+1}}\int \frac{d\mu}{2}\mu P_\ell(\mu)\Theta(\mu) = \frac{k(\ell+1)}{(2\ell+1)}\underbrace{\frac{1}{(-i)^{\ell+1}}\int \frac{d\mu}{2}P_{\ell+1}(\mu)\Theta(\mu)}_{\Theta_{\ell+1}} + \frac{k\ell}{(2\ell+1)}\underbrace{\frac{1}{(-i)^{\ell+1}}\int \frac{d\mu}{2}P_{\ell-1}(\mu)\Theta(\mu)}_{\frac{\Theta_{\ell-1}}{(-i)^2}} =$$

$$= \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1} + \frac{k\ell}{(2\ell+1)}\frac{1}{(-i)^2}\Theta_{\ell-1} = \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1} - \frac{k\ell}{2\ell+1}\Theta_{\ell-1}$$

Summarizing integrated Boltzmann equation ($\ell > 2$)

$$\boxed{\dot{\Theta}_\ell - \frac{k\ell}{2\ell+1}\Theta_{\ell-1} + \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1} = \dot{\tau}\Theta_\ell}$$

Look at orders of magnitude (for the moment neglect $\Theta_{\ell+1}$)

$$\dot{\Theta}_\ell \sim \frac{\Theta_\ell}{\eta} \ll \dot{\tau}\Theta_\ell \sim \tau\frac{\Theta_\ell}{\eta} \qquad \text{since } \tau \gg 1$$

$$\frac{k\ell}{2\ell+1}\Theta_{\ell-1} \sim \frac{k}{2}\Theta_{\ell-1}$$

so $\dot{\Theta}_\ell - \frac{k\ell}{2\ell+1}\Theta_{\ell-1} \sim -\frac{k}{2}\Theta_{\ell-1} \sim \dot{\tau}\Theta_\ell \sim \frac{\tau\Theta_\ell}{\eta} \Rightarrow \frac{k\Theta_{\ell-1}}{2} \sim \frac{\tau\Theta_\ell}{\eta}$

$$\Rightarrow \boxed{\Theta_\ell \sim \frac{k\eta}{2\tau}\Theta_{\ell-1}}$$

For horizon-sized modes $k\eta \sim 1 \Rightarrow \boxed{\Theta_\ell \ll \Theta_{\ell-1}}$

So all $\ell > 2$ modes are $\ll \Theta_0, \Theta_1$ $\rightarrow$ can neglect all modes except monopole and dipole $\rightarrow$ fluid approximation

Physical meaning

observer sees photons arriving from $\lambda_{MFP} \sim \frac{\eta}{\tau}$, for $k\eta \sim 1$ perturbation wavelength much larger, so see photons with same temperature. For small wavelength $k\eta \sim \tau$, this is smaller than the damping scale, so only monopole and dipole survive

Summary: In the tightly coupled regime, only Θ_0 and Θ_1 survive

So now get equations for Θ_0 and Θ_1 starting from:

$$\dot\Theta + ik\mu\Theta = -\dot\Phi - ik\mu\Psi - \dot\tau\left[\Theta_0 - \Theta + \mu v_b\right] \quad \text{with} \quad \begin{cases} \Theta_0 = \int \frac{d\mu}{2}\,\Theta(\mu) \\ \Theta_1 = i\int \frac{d\mu}{2}\,\mu\,\Theta(\mu) \\ \Theta_2 = 0 \to \int d\mu\,\mu^2\Theta \approx \frac{\Theta_0}{3} \end{cases}$$

Monopole

$$\underbrace{\int\frac{d\mu}{2}\dot\Theta}_{\dot\Theta_0} + ik\underbrace{\int\frac{d\mu}{2}\mu\Theta}_{\frac{\Theta_1}{i}} = -\dot\Phi\underbrace{\int\frac{d\mu}{2}}_{\frac{\mu}{2}\big|_{-1}^{1}=1} - ik\Psi\underbrace{\int\frac{d\mu}{2}\mu}_{\frac{\mu^2}{4}\big|_{-1}^{1}=0} - \dot\tau\Theta_0\underbrace{\int\frac{d\mu}{2}}_{\frac{\mu}{2}\big|_{-1}^{1}=1} + \dot\tau\underbrace{\int\frac{d\mu}{2}\Theta}_{\Theta_0} - \dot\tau v_b\underbrace{\int\frac{d\mu}{2}\mu}_{\frac{\mu^2}{4}\big|_{-1}^{1}=0}$$

$$\Rightarrow \dot\Theta_0 + k\Theta_1 = -\dot\Phi - \cancel{\dot\tau\Theta_0} + \cancel{\dot\tau\Theta_0} \quad \Rightarrow \quad \dot\Theta_0 + k\Theta_1 = -\dot\Phi$$

Dipole

$$i\underbrace{\int\frac{d\mu}{2}\mu\dot\Theta}_{\dot\Theta_1} - k\underbrace{\int\frac{d\mu}{2}\mu^2\Theta}_{\approx\frac{\Theta_0}{3}} = -i\dot\Phi\underbrace{\int\frac{d\mu}{2}\mu}_{\frac{\mu^2}{4}\big|_{-1}^{1}=0} + k\Psi\underbrace{\int\frac{d\mu}{2}\mu^2}_{\frac{\mu^3}{6}\big|_{-1}^{1}=\frac{1}{3}} - i\dot\tau\Theta_0\underbrace{\int\frac{d\mu}{2}\mu}_{\frac{\mu^2}{4}\big|_{-1}^{1}=0} + i\dot\tau\underbrace{\int\frac{d\mu}{2}\mu\Theta}_{\dot\tau\Theta_1} - i\dot\tau v_b\underbrace{\int\frac{d\mu}{2}\mu^2}_{\frac{\mu^3}{6}\big|_{-1}^{1}=\frac{1}{3}}$$

$$\Rightarrow \dot\Theta_1 - \frac{k\Theta_0}{3} = \frac{k\Psi}{3} + \dot\tau\left[\Theta_1 - \frac{iv_b}{3}\right]$$

So equations for monopole and dipole in tightly-coupled limit:

$$\boxed{\begin{aligned} &\dot\Theta_0 + k\Theta_1 = -\dot\Phi \\ &\dot\Theta_1 - \frac{k\Theta_0}{3} = \frac{k\Psi}{3} + \dot\tau\left[\Theta_1 - \frac{iv_b}{3}\right] \end{aligned}}$$

Need also equations for baryon fluid
$$\begin{cases} \dot\delta_b + ikv_b = -3\dot\Phi \\ \dot v_b + \frac{\dot a}{a}v_b = -ik\Psi + \frac{\dot\tau}{R}\left[v_b + 3i\Theta_1\right] \end{cases}$$

Consider velocity equation

$$\dot{v}_b + \frac{\dot{a}}{a} v_b = -ik\psi + \frac{\dot{\tau}}{R}\left[v_b + 3i\Theta_1\right]$$

Rewrite it as $\rightarrow$ $$v_b = -3i\Theta_1 + \underbrace{\frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a} v_b + ik\psi\right]}_{\text{Suppressed by } \tau^{-1} \ll 1 \text{ relative to } -3i\Theta_1}$$

So to lowest order $v_b \simeq -3i\Theta_1$ (i.e. $v_b \approx v_\gamma$) and we can use this lowest-order expansion everywhere

$$v_b = -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a} v_b + ik\psi\right] \rightarrow -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[3i\dot{\Theta}_1 - \frac{\dot{a}}{a} 3i\Theta_1 + ik\psi\right]$$

$$\rightarrow v_b \simeq -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[-3i\dot{\Theta}_1 - 3i\frac{\dot{a}}{a}\Theta_1 + ik\psi\right]$$

$\hookrightarrow$ plug into dipole equation $$\dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{iv_b}{3}\right]$$

$$\Theta_1 - \frac{iv_b}{3} \simeq \Theta_1 - \Theta_1 - \frac{i}{3}\frac{R}{\dot{\tau}}\left[-3i\Theta_1 - 3i\frac{\dot{a}}{a}\Theta_1 + ik\psi\right] = \cancel{\Theta_1} - \cancel{\Theta_1} - \frac{R}{\dot{\tau}}\left[\dot{\Theta}_1 + \frac{\dot{a}}{a}\Theta_1 - \frac{k\psi}{3}\right] =$$

$$= -\frac{R}{\dot{\tau}}\left[\dot{\Theta}_1 + \frac{\dot{a}}{a}\Theta_1 - \frac{k\psi}{3}\right]$$

$$\dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{iv_b}{3}\right] \Rightarrow \dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\psi}{3} - \dot{\tau}\frac{R}{\dot{\tau}}\left[\dot{\Theta}_1 + \frac{\dot{a}}{a}\Theta_1 - \frac{k\psi}{3}\right] =$$

$$= -R\dot{\Theta}_1 - \frac{\dot{a}}{a}R\Theta_1 + \frac{k\psi}{3}(1+R)$$

$$\Rightarrow \dot{\Theta}_1 - \frac{k\Theta_0}{3} = -R\dot{\Theta}_1 - \frac{\dot{a}}{a}R\Theta_1 + \frac{k\psi}{3}(1+R)$$

$$\Rightarrow \dot{\Theta}_1(1+R) + \frac{\dot{a}}{a}R\Theta_1 - \frac{k\Theta_0}{3} = \frac{k\psi}{3}(1+R)$$

$$\Rightarrow \boxed{\dot{\Theta}_1 + \frac{\dot{a}}{a}\frac{R}{1+R}\Theta_1 - \frac{k}{3(1+R)}\Theta_0 = \frac{k\Psi}{3}}$$

So now we have two coupled 1st order equations for Θ_0 and Θ_1

$$\begin{cases} \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} & (1) \\ \dot{\Theta}_1 + \frac{\dot{a}}{a}\frac{R}{1+R}\Theta_1 - \frac{k}{3(1+R)}\Theta_0 = \frac{k\Psi}{3} & (2) \end{cases}$$

Turn this into 2nd order equation for Θ_0 by differentiating (1), using (2) to eliminate $\dot{\Theta}_1$

$$\ddot{\Theta}_0 + k\dot{\Theta}_1 = -\ddot{\Phi} \quad \text{with} \quad \dot{\Theta}_1 = -\frac{\dot{a}}{a}\frac{R}{1+R}\Theta_1 + \frac{k}{3(1+R)}\Theta_0 + \frac{k\Psi}{3}$$

$$\Rightarrow \ddot{\Theta}_0 + k\left(-\frac{\dot{a}}{a}\frac{R}{1+R}\Theta_1 + \frac{k}{3(1+R)}\Theta_0 + \frac{k\Psi}{3}\right) = -\ddot{\Phi}$$

$$\Rightarrow \cancel{\ddot{\Theta}_0 + k \frac{k\Psi}{3}} \quad \text{then use: } \Theta_1 = -\frac{\dot{\Phi}}{k} - \frac{\dot{\Theta}_0}{k}$$

$$\ddot{\Theta}_0 + k\left[-\frac{\dot{a}}{a}\left(-\frac{\dot{\Phi}}{k} - \frac{\dot{\Theta}_0}{k}\right)\frac{R}{1+R} + \frac{k}{3(1+R)}\Theta_0 + \frac{k\Psi}{3}\right] = -\ddot{\Phi}$$

$$\Rightarrow \ddot{\Theta}_0 + \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0 + \frac{k^2}{3(1+R)}\Theta_0 = -\frac{k^2\Psi}{3} - \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Phi} - \ddot{\Phi}$$

Define sound speed of the fluid as

$$c_s \equiv \sqrt{\frac{1}{3(1+R)}} = \sqrt{\frac{1}{3\left(1+\frac{3\rho_b}{4\rho_\gamma}\right)}} \quad \begin{array}{l} \xrightarrow{a\to 0 \quad \rho_b \to 0} \frac{1}{\sqrt{3}} \\ \xrightarrow[a\to\infty \quad \rho_\gamma \to 0]{} 0 \end{array}$$

$R \equiv \frac{3\rho_b}{4\rho_\gamma}$

With this definition $\zeta \downarrow$ as $\beta \uparrow$ because baryons make fluid heavier

$$\boxed{\ddot{\Theta}_0 + \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0 + k^2 c_s^2 \Theta_0 = -\frac{k^2}{3}\Psi - \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Phi} - \ddot{\Phi} \equiv F(k,\eta)}$$

where driving force

$$F(k,\eta) = -\frac{k^2}{3}\Psi - \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Phi} - \ddot{\Phi}$$

This is the more complete version of the cartoon equation

$$\ddot{\Theta}_0 + k^2 c_s^2 \Theta_0 = F$$

by additional damping term $\frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0$

$$\boxed{\ddot{\Theta}_0 + \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0 + k^2 c_s^2 \Theta_0 = F(k,\eta)}$$

Note Φ enters in the right in a way very similar to Θ_0 on the left

Alternate form of equation

$$\boxed{\left\{\frac{d^2}{d\eta^2} + \frac{\dot{R}}{1+R}\frac{d}{d\eta} + k^2 c_s^2\right\}\left[\Theta_0 + \Phi\right] = \frac{k^2}{3}\left[\frac{1}{1+R}\Phi - \Psi\right]}$$

Tightly coupled solutions

$\boxed{}$ = 2nd order ODE, need 2 solutions to homogeneous equation

to construct Green's function

To get some insight neglect damping

R small; modes within horizon $k^2\eta^2 \gg 1$

damping term $\simeq \frac{\dot{R}}{1+R}\frac{d}{d\eta}(\Theta_0 + \Phi) \sim \frac{R(\Theta_0+\Phi)}{\eta^2}$

pressure term $\sim k^2 c_s^2(\Theta_0 + \Phi)$

$$\frac{\text{pressure}}{\text{damping}} \sim \frac{k^2 c_s^2 \eta^2}{R} \gg 1 \text{ if}$$

We expect pressure to set up oscillations on scale much smaller than damping due to expansion of Universe

- Approximate (oscillating) equation

$$\left(\frac{d^2}{d\eta^2}+k^2c_s^2\right)\left[\Theta_0+\Phi\right]=\frac{k^2}{3}\left[\frac{\Phi}{1+R}-\Psi\right]$$

Homogeneous equation

$$\left[\frac{d^2}{d\eta^2}+k^2c_s^2(\eta)\right](\Theta_0+\Phi)=0$$

- solutions to the homogeneous equation

$$S_1(k,\eta)=\sin\left[kr_s(\eta)\right] \qquad S_2(k,\eta)=\cos\left[kr_s(\eta)\right]$$

Sound horizon $\quad r_s(\eta)\equiv\int_0^\eta d\eta' c_s(\eta')$

Now can construct full solution. In source term drop R $(1+R\sim 1)$ so it's just $\frac{k^2}{3}(\Phi-\Psi)$, but don't drop R in sin/cos argument

- $$\Theta_0(\eta)+\Phi(\eta)\simeq C_1S_1(\eta)+C_2S_2(\eta)+\frac{k^2}{3}\int_0^\eta d\eta'\left[\Phi(\eta')-\Psi(\eta')\right]\frac{\boxed{S_1(\eta')S_2(\eta)-S_1(\eta)S_2(\eta')}}{\boxed{S_1(\eta')\dot{S}_2(\eta')-\dot{S}_1(\eta')S_2(\eta')}}$$

$$S_1(\eta')S_2(\eta)-S_1(\eta)S_2(\eta')=\sin[kr_s(\eta')]\cos[kr_s(\eta)]-\sin[kr_s(\eta)]\cos[kr_s(\eta')]=$$
$$=\sin\left[k(r_s(\eta)-r_s(\eta'))\right]$$ (just a simple trigonometric identity)

$$S_1(\eta')\dot{S}_2(\eta')-\dot{S}_1(\eta')S_2(\eta')=?$$

$$\dot{S}_1(\eta')=+k\frac{dr_s}{d\eta}\cos[kr_s(\eta')] \qquad \frac{dr_s}{d\eta}=\frac{d}{d\eta}\int_0^\eta d\eta' c_s(\eta')=c_s \longrightarrow =+kc_s\cos[kr_s(\eta')]$$

- and similarly $\dot{S}_2(\eta')=-kc_s\sin[kr_s(\eta')]$

So $S_1(\eta')\dot{S}_2(\eta')-\dot{S}_1(\eta')S_2(\eta')=-kc_s(\eta')\sin^2[kr_s(\eta')]-kc_s(\eta')\cos^2[kr_s(\eta')]=-kc_s(\eta')$

but in the limit we are working in

$R \ll 1 \Rightarrow 1+R \sim 1 \Rightarrow c_s \simeq \frac{1}{\sqrt{3}}$

$\Rightarrow -k c_s(\eta') \simeq -\frac{k}{\sqrt{3}}$

Putting everything together

$$\frac{S_1(\eta')S_2(\eta)-S_1(\eta)S_2(\eta')}{S_1(\eta')\dot{S}_2(\eta')-\dot{S}_1(\eta')S_2(\eta')} \simeq \frac{-\sin[k(r_s(\eta)-r_s(\eta'))]}{-k/\sqrt{3}} = \frac{\sqrt{3}\sin[k(r_s(\eta)-r_s(\eta'))]}{k}$$

$$\Theta_0(\eta)+\Phi(\eta) \simeq C_1\sin[k r_s(\eta)] + C_2\cos[k r_s(\eta)]$$
$$+\frac{k}{\sqrt{3}}\int_0^{\eta} d\eta' \, [\Phi(\eta')-\Psi(\eta')] \sin[k(r_s(\eta)-r_s(\eta'))]$$

Now just need to fix C_1 and C_2. Initial conditions:

$\Theta_0(k,\eta_*) = \frac{2\Phi_p}{3} = \text{const}$

$\Rightarrow C_1 = 0 \quad , \quad C_2 = \Theta_0(0)+\Phi(0) \qquad$ as $\sin[k r_s(\eta)] \xrightarrow{\eta\to 0} 0$

$\cos[k r_s(\eta)] \xrightarrow{\eta\to 0} 1$

$\Rightarrow$ Full expression of anisotropy in the tightly coupled limit

$$\boxed{\Theta_0(\eta)+\Phi(\eta) = [\Theta_0(0)+\Phi(0)]\cos(k r_s(\eta)) + \frac{k}{\sqrt{3}}\int_0^{\eta} d\eta' \, \underbrace{[\Phi(\eta')-\Psi(\eta')]} \sin[k(r_s(\eta)-r_s(\eta'))]}$$

we get it from elsewhere (e.g. previous chapter)

Noteworthy feature of solution:

- agrees well with full numerical solution for location of acoustic peaks and height of first (undamped) peaks
- clearly separates solving for Φ, Ψ, then computing their effect on anisotropies
- shows that inflation excites cosine mode (without inflation by causality should have no perturbations for $k\eta \ll 1$ and therefore ~~cos~~ sine mode should be excited)
- when cos term dominates, peaks at

$$\cos(k_p r_s) = \pm 1 \longrightarrow k_p r_s = n\pi \longrightarrow k_p = \frac{n\pi}{r_s}$$

- Reduced full set of Θ_l equations to just one!

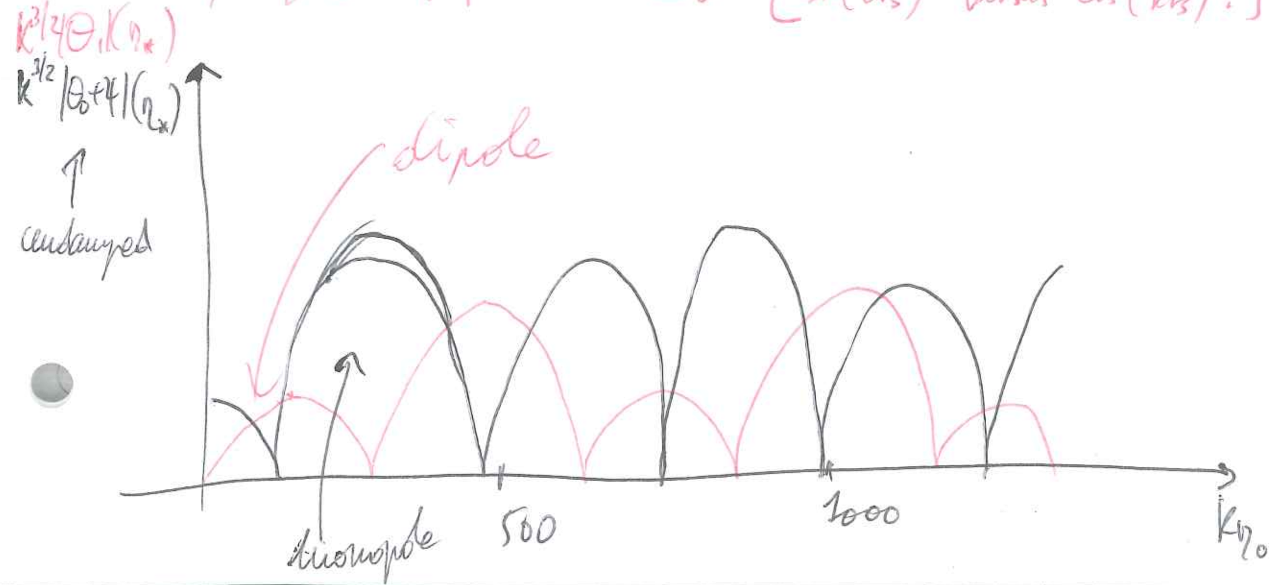
Beyond Θ_0, also Θ_1 important. Recall

$$\dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \longrightarrow \Theta_1 = -\frac{\dot{\Phi}}{k} - \frac{\dot{\Theta}_0}{k} \quad \text{involves } \dot{\Theta}_0$$

$\cos \to -\sin$

$$\Theta_1(\eta) \simeq \boxed{\frac{1}{\sqrt{3}}\left[\Theta_0(0)+\Phi(0)\right]\sin\left[k r_s(\eta)\right]} - \frac{k}{3}\int_0^{\eta} d\eta' \left[\Phi(\eta')-\Psi(\eta')\right]\cos\left[k\left(r_s(\eta)-r_s(\eta')\right)\right]$$

completely out of phase with Θ_0 [$\sin(kr_s)$ versus $\cos(kr_s)$!]



Diffusion damping

Recall equation for photon distribution moments

$$\begin{cases} \dot{\Theta}_\ell - \frac{k\ell}{2\ell+1}\Theta_{\ell-1} + \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1} = \dot{\tau}\Theta_\ell \\ \dot{\Theta}_0 + k\Theta_1 = -\dot{\Phi} \\ \dot{\Theta}_1 - \frac{k\Theta_0}{3} = \frac{k\Psi}{3} + \dot{\tau}\left[\Theta_1 - \frac{i v_b}{3}\right] \end{cases}$$

Diffusion damping characterized by small but non-negligible Θ_2

We now care only about very small scales, where Φ, Ψ are negligible as they are damped after horizon crossing $\left[\Phi \sim \frac{\sin/\cos(k\eta)}{(k\eta)^3}\right]$

$\Downarrow$

$$\begin{cases} \dot{\Theta}_0 + k\Theta_1 = 0 \\ \dot{\Theta}_1 + \boxed{k\left(\frac{2}{3}\Theta_2\right.}\left.-\frac{1}{3}\Theta_0\right) = \dot{\tau}\left[\Theta_1 - \frac{i v_b}{3}\right] \\ \boxed{\dot{\Theta}_2 - \frac{2k}{5}\Theta_1 = \frac{9}{10}\dot{\tau}\Theta_2} \end{cases}$$

$\boxed{} = \frac{k(\ell+1)}{2\ell+1}\Theta_{\ell+1}$ with $\ell = 1$

$\rightarrow$ by doing $\int \times \frac{d\mu}{2} P_2(\mu)$

Θ_ℓ, $\ell \geq 3$ neglected as higher moments are suppressed by $\frac{1}{\dot{\tau}}$

We also need equation for v_b

Start from earlier

$$v_b \simeq -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a}v_b + ik\Psi\right] \simeq -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a}v_b\right]$$

neglect Ψ!

$$\Rightarrow 3i\Theta_1 + v_b \simeq \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a}v_b\right]$$

$\uparrow$ equation for v_b

Use WKB-like approximation given that damping is a high-frequency phenomenon (many damping in $\sim \frac{1}{H}$)

$$v_b \propto e^{i\int \omega d\eta}$$

and similarly for $\theta_0, \theta_1, \theta_2$

In the tightly coupled limit $\omega \simeq k c_s$ real. Damping corresponds to $\mathrm{Im}(\omega)$

$$v_b \propto e^{i\int d\eta\, \omega(\eta)} \longrightarrow \dot{v}_b = i\omega e^{i\int d\eta\, \omega(\eta)} = i\omega v_b \gg \frac{\dot{a}}{a} v_b$$

Damping frequency $\gg$ "Expansion frequency" as $\frac{\dot{a}}{a} \propto \frac{1}{\eta}$, $\omega \propto k$

$$3i\theta_1 + v_b = \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a} v_b\right] \simeq \frac{R}{\dot{\tau}}\dot{v}_b = \frac{R}{\dot{\tau}} i\omega v_b$$

$$\Rightarrow v_b + 3i\theta_1 \simeq \frac{R}{\dot{\tau}} i\omega v_b \Longrightarrow v_b\left[1 - \frac{i\omega R}{\dot{\tau}}\right] \simeq -3i\theta_1$$

$$\Rightarrow v_b \simeq -3i\theta_1\left[1 - \frac{i\omega R}{\dot{\tau}}\right]^{-1} \simeq -3i\theta_1\left[1 + \frac{i\omega R}{\dot{\tau}} - \left(\frac{\omega R}{\dot{\tau}}\right)^2\right]$$

$(1-x)^{-1} \simeq 1 + x + x^2 + \mathcal{O}(x^3)$

Expand to $\frac{1}{\dot{\tau}^2}$ as $(v_b + 3i\theta_1)$ multiplied by $\dot{\tau}$ in $\dot{\theta}_1$ equation

Similarly reduce θ_2 equation

$$\cancel{\dot{\theta}_2} - \frac{2k}{5}\theta_1 = \frac{9}{10}\dot{\tau}\theta_2 \xrightarrow[\dot{\tau} \gg \frac{1}{\eta}]{\dot{\theta}_2 \ll \frac{9}{10}\dot{\tau}\theta_2} \theta_2 = -\frac{2k}{5}\frac{10}{9\dot{\tau}}\theta_1 = -\frac{4k}{9}\theta_1$$

So $\theta_2 = \frac{4k}{9\dot{\tau}}\theta_1$ confirming that higher moments are

indeed supressed by $\frac{k}{\dot{\tau}} \ll 1 \Longrightarrow \dot{\tau} \gg k$

~~Zero~~ Same with equation for θ_0

$$\dot{\theta}_0 + k\theta_1 = 0 \qquad \theta_0 \propto e^{i\int d\eta\, \omega(\eta)} \Rightarrow \dot{\theta}_0 = i\omega\theta_0$$

$$\Rightarrow i\omega\theta_0 + k\theta_1 = 0 \Rightarrow i\omega\theta_0 = -k\theta_1$$

Overall we get

$$\begin{cases} i\omega\theta_0 \simeq -k\theta_1 & \longrightarrow -\frac{1}{3}\theta_0 \simeq +\frac{k}{3i\omega}\theta_1 \\ \theta_2 \simeq -\frac{4k}{9\dot{\tau}}\theta_1 & \longrightarrow \frac{2}{3}\theta_2 \simeq -\frac{8k}{27\dot{\tau}}\theta_1 \\ v_b \simeq -3i\theta_1\left[1+\frac{i\omega R}{\dot{\tau}}-\left(\frac{\omega R}{\dot{\tau}}\right)^2\right] & \longrightarrow -\frac{iv_b}{3} \simeq -\theta_1\left[1+\frac{i\omega R}{\dot{\tau}}-\left(\frac{\omega R}{\dot{\tau}}\right)^2\right] \end{cases}$$

Put all these inside equation for θ_1

$$\dot{\theta}_1 + k\left(\frac{2}{3}\theta_2 - \frac{1}{3}\theta_0\right) = \dot{\tau}\left(\theta_1 - \frac{iv_b}{3}\right) \qquad \theta_1 \propto e^{i\int d\eta\, \omega(\eta)} \Rightarrow \dot{\theta}_1 = i\omega\theta_1$$

$$\Rightarrow i\omega\cancel{\theta_1} - \frac{8k^2}{27\dot{\tau}}\cancel{\theta_1} + \frac{k^2}{3i\omega}\cancel{\theta_1} = \dot{\tau}\left(1-\left[1+\frac{i\omega R}{\dot{\tau}}-\left(\frac{\omega R}{\dot{\tau}}\right)^2\right]\right)\cancel{\theta_1}$$

$$\Rightarrow i\omega - \frac{8k^2}{27\dot{\tau}} + \frac{k^2}{3i\omega} = \dot{\tau}\left(1-\left[1+\frac{i\omega R}{\dot{\tau}}-\left(\frac{\omega R}{\dot{\tau}}\right)^2\right]\right)$$

$$\Rightarrow i\omega - \frac{8k^2}{27\dot{\tau}} + \frac{k^2}{3i\omega} = \cancel{\dot{\tau}} - \cancel{\dot{\tau}} - i\omega R + \frac{(\omega R)^2}{\dot{\tau}}$$

$\times -i\omega$

$$\Rightarrow \omega^2 + \frac{8ik^2}{27\dot{\tau}}\omega - \frac{k^2}{3} = -\omega^2 R - i\frac{(\omega R)^2}{\dot{\tau}}\omega$$

$$\Rightarrow \boxed{\omega^2(1+R) - \frac{k^2}{3} + \frac{i\omega}{\dot{\tau}}\left[\omega^2R^2 + \frac{8k^2}{27}\right] = 0}$$

Dispersion relation

Consider no damping, ~~i.e.~~ neglect $\dot{\tau}$ term

$$\Rightarrow \omega^2(1+R) = \frac{k^2}{3} \longrightarrow \omega^2 = \frac{k^2}{3(1+R)}$$

standard dispersion relation

$\omega^2 = c^2 k^2$ with $c = \sqrt{\frac{1}{3(1+R)}}$

With damping there is an imaginary part

$$\omega = \omega_0 + \delta\omega \qquad \omega_0^2 = c_s^2 k^2 \qquad \omega_0 = c_s k$$

$$\omega = c_s k + \delta\omega \qquad \delta\omega \text{ first order in } \frac{1}{\dot{\tau}}$$

↳ re-insert into dispersion relation, look at 1st order terms

$$\underbrace{\cancel{c_s^2 k^2 (1+R)}}_{=\frac{k^2}{3}} + 2c_s k(1+R)\delta\omega - \cancel{\frac{k^2}{3}} + \frac{i c_s k}{\dot{\tau}}\left[c_s^2 k^2 R^2 + \frac{8k^2}{27}\right] = 0$$

since $c_s^2 = \frac{1}{3(1+R)}$

$$\Rightarrow 2\cancel{c_s k}(1+R)\delta\omega + \frac{i\cancel{c_s k}}{\dot{\tau}}\left[c_s^2 k^2 R^2 + \frac{8k^2}{27}\right] = 0$$

$$\Rightarrow \boxed{\delta\omega \simeq -\frac{ik^2}{2(1+R)\dot{\tau}}\left[c_s^2 R^2 + \frac{8}{27}\right]}$$

Recall $\Theta_0, \Theta_1 \propto \exp\left[i\int d\eta\, \omega(\eta)\right] = \exp\left\{i\int d\eta \left[\omega(\eta) + \delta\omega(\eta)\right]\right\}$

So $\omega \simeq \omega_0 + \delta\omega = c_s k + \delta\omega \rightarrow i\omega \simeq i c_s(\eta) k + i\delta\omega(\eta)$

$$\Theta_0, \Theta_1 \propto \exp\left\{ik\int d\eta\, c_s(\eta)\right\} \exp\left\{-\frac{k^2}{k_D^2}\right\}$$

where we defined

$$\frac{1}{k_D^2(\eta)} \equiv \int_0^\eta \frac{d\eta'}{6(1+R) n_e \sigma_T a(\eta')}\left[\frac{R^2}{(1+R)} + \frac{8}{9}\right]$$

Where does this come from?

$$\Theta_0, \Theta_1 \propto e^{i\int d\eta\, \delta\omega(\eta)}$$

$$\dot{\tau} = -a n_e \sigma_T$$

$$c_s^2 = \frac{1}{3(1+R)}$$

$$i\,\delta\omega(\eta) = \frac{k^2}{2(1+R)\dot{\tau}}\left[c_s^2 R^2 + \frac{8}{27}\right] =$$

$$= -\frac{k^2}{2(1+R)a n_e \sigma_T}\left[\frac{R^2}{3(1+R)} + \frac{8}{27}\right] = -\frac{k^2}{6(1+R)n_e\sigma_T a}\left[\frac{R^2}{(1+R)} + \frac{8}{9}\right]$$

$$\int d\eta\, i\,\delta\omega(\eta) = -k^2\int_0^{\eta} d\eta' \frac{1}{6(1+R)n_e\sigma_T a(\eta')}\left[\frac{R^2}{(1+R)} + \frac{8}{9}\right] \equiv -\frac{k^2}{k_D^2}$$

So $\frac{1}{k_D} \sim \sqrt{\frac{\eta}{n_e \sigma_T a}}$ as we had guessed earlier

To understand k_D, work in prerecombination limit, all electrons ionized except in Helium.

$$n_e \sigma_T a \simeq 2.3\times10^{-5}\,\mathrm{Mpc}^{-1}\,\Omega_b h^2 a^{-2}\left(1-\frac{Y_p}{2}\right)$$

$$\rightarrow k_D^{-2} \simeq 3.1\times10^{6}\,\mathrm{Mpc}^2\, a^{5/2} f_D\!\left(\frac{a}{a_{eq}}\right)(\Omega_b h^2)^{-1}\left(1-\frac{Y_p}{2}\right)^{-1}$$

with $f_D \rightarrow 1$ as $\frac{a}{a_{eq}} \gg 1$

as $\Omega_b \uparrow$ $k_D \uparrow$ $k_D^{-1} \downarrow$