

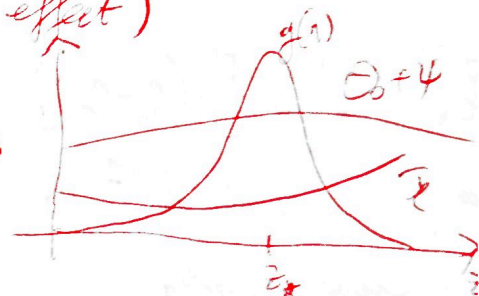
• (3) $\int_0^{p_0} d\eta e^{-\tau} [\Psi(k, \eta) - \Phi(k, \eta)] \bar{J}_\ell [k(\eta_0 - \eta)]$

$e^{-\tau}$ non-negligible only when $\tau \lesssim 1$, well after recombination, when matter dominates and Ψ, Φ constant (both on large & small scales) so $\Psi, \Phi \sim 0$

(3) negligible to zeroth order (but gives ~10% corrections through early μ and like integrated Sachs-Wolfe effect)

• (2) ~~negligible~~

at a certain l, k



• (1) $\int_0^{p_0} d\eta g(\eta) [\Theta_0(k, \eta) + \Psi(k, \eta)] \bar{J}_\ell [k(\eta_0 - \eta)]$

$g(\eta)$ sharply peaked around η_*

$$\approx [\Theta_0(k, \eta_*) + \Psi(k, \eta_*)] \bar{J}_\ell [k(\eta_0 - \eta_*)] \int_0^{p_0} d\eta g(\eta) = [\Theta_0(k, \eta_*) + \Psi(k, \eta_*)] \bar{J}_\ell [k(\eta_0 - \eta_*)]$$

• (2) Similar considerations

$$\int_0^{p_0} d\eta g(\eta) \frac{i v_\ell(k, \eta)}{k} \frac{d}{d\eta} \bar{J}_\ell [k(\eta_0 - \eta)] \approx -\frac{i v_\ell(k, \eta_*)}{k} \frac{d}{d\eta} \bar{J}_\ell [k(\eta_0 - \eta_*)] \int_0^{p_0} d\eta g(\eta) =$$

$$= -\frac{i v_\ell(k, \eta_*)}{k} \frac{d}{d\eta} \bar{J}_\ell [k(\eta_0 - \eta_*)] \Big|_{\eta_*} \approx 3\Theta_0(k, \eta_*) \left(\bar{J}_{\ell-1} [k(\eta_0 - \eta_*)] - \frac{(\ell+1) \bar{J}_\ell [k(\eta_0 - \eta_*)]}{k(\eta_0 - \eta_*)} \right)$$

$v_\ell \approx -3\Theta_\ell, \frac{dv_\ell}{dk} = -\bar{J}_{\ell-1} - \frac{\ell+1}{k} \bar{J}_\ell, \frac{d}{d[k(\eta_0 - \eta)]} = -\frac{1}{k} \frac{d}{d\eta}$

In reality also need to account for damping:

$$e^{-\frac{k^2}{k_0(\eta_*)^2}} \rightarrow \int d\eta g(\eta) e^{-\frac{k^2}{k_0(\eta)^2}}$$

In summary

$$\Theta_\ell(k, \eta_0) \approx [\Theta_0(k, \eta_0) + \Psi(k, \eta_*)] \mathcal{Y}_\ell[k(\eta_0 - \eta_*)] \\ + 3\Theta_1(k, \eta_*) \left(\mathcal{Y}_{\ell-1}[k(\eta_0 - \eta_*)] - \frac{(\ell+1) \mathcal{Y}_\ell[k(\eta_0 - \eta_*)]}{k(\eta_0 - \eta_*)} \right) \\ + \int_{\eta_0}^{\eta_*} d\eta e^{-\tau} [\dot{\Psi}(k, \eta) - \dot{\Phi}(k, \eta)] \mathcal{Y}_\ell[k(\eta_0 - \eta)]$$

So to compute Θ_ℓ today we need to know Θ_0, Θ_1, Ψ at recombination
 $\mathcal{Y}_\ell(k(\eta_0 - \eta_*)) \mathcal{Y}_\ell[k(\eta_0 - \eta_*)]$ determines how much anisotropy on an angular scale l^{-1} is contributed by a plane wave with wavenumber k

On small scales

$$\lim_{l \rightarrow \infty} \mathcal{Y}_\ell(x) = \frac{1}{l} \left(\frac{x}{l}\right)^{l-\frac{1}{2}} \ll 1 \text{ for } l \gg x \text{ i.e. } l \gg k(\eta_0 - \eta_*) \approx k\eta_0$$

so $\Theta_\ell(k, \eta_0) \approx 0$ for $l \gg k\eta_0$



small angular scales see little anisotropy from large-wavelength perturbation
 and conversely ~~or equivalently~~ angular scales $\approx \frac{1}{k\eta_0}$ get little contribution from such a perturbation

So perturbation with wavenumber k mostly contributes to anisotropies on angular scales $l \sim k\eta_0 \rightarrow k \sim \frac{l}{\eta_0}$

Now we need to connect the abstract $\Theta_\ell(k, \eta_0)$ to observed ℓ
 How do we characterize the temperature field today?

$$T(\bar{x}, \hat{p}, \eta) = T(\eta) [1 + \Theta(\bar{x}, \hat{p}, \eta)]$$

T defined at every point in space and time, but we can only

measure \hat{T} here (\bar{x}_0) and now $(\eta_0) = T(\bar{x}_0, \hat{p}, \eta_0)$

Anisotropies encoded in direction of incoming photons \hat{p}

CMB maps labeled not as $\hat{p}_x, \hat{p}_y, \hat{p}_z$, but in polar coordinates (θ, φ) such that $\hat{p}_z = \cos\theta$, $\hat{p}_x = \sin\theta \cos\varphi$, $\hat{p}_y = \sin\theta \sin\varphi$
 let's stick with \hat{p} , expand Θ in spherical harmonics (since observed on the sky)

$$\Theta(\bar{x}, \hat{p}, \eta) = \sum_{l=1}^{\infty} \sum_{m=-l}^l a_{lm}(\bar{x}, \eta) Y_{lm}(\hat{p})$$



Y_{lm} : generalize Fourier transform on sphere, complete set of eigenfunctions replacing $e^{i\vec{k}\cdot\vec{x}}$ for flat space

So all information contained in full $T(\hat{p})$ is also contained in a_{lm}

Given a certain angular resolution σ_θ that corresponds to

$l_{\max} \sim \frac{\pi}{\sigma_\theta}$ given by total number of recoverable a_{lm}

$$\sum_{l=0}^{l_{\max}} (2l+1) = (l_{\max} + 1)^2 = \frac{4\pi \text{ rad}^2}{\sigma_\theta [\text{rad}^2]} \quad (\text{e.g. } \cos\theta \sigma_\theta \approx \pi^\circ, l_{\max} \approx 3)$$

So basically the observables are the a_{lm} s: how can they be related to $\Theta(\bar{k}, \eta_0)$? Use orthogonality property

of spherical harmonics:

$$\int d\Omega Y_{lm}(\hat{p}) Y_{l'm'}^*(\hat{p}) = \delta_{ll'} \delta_{mm'}$$

$$\Theta(\bar{x}, \hat{p}, \eta) = \sum_{l=1}^{\infty} \sum_{m=-l}^l a_{lm}(\bar{x}, \eta) Y_{lm}(\hat{p})$$

$$\int d\Omega Y_{lm}(\hat{p}) Y_{l'm'}^*(\hat{p}) = \delta_{ll'} \delta_{mm'}$$

⇓

$$a_{lm}(\bar{k}, \eta) \stackrel{?}{=} \int d\Omega Y_{lm}^*(\hat{p}) \Theta(\bar{k}, \hat{p}, \eta) = \int d\Omega \sum_{l'm'} a_{l'm'} Y_{l'm'} Y_{lm}^* =$$

$$= \int d\Omega \sum_{l'm'} \delta_{ll'} \delta_{mm'} a_{l'm'} = a_{lm} \quad \text{QED}$$

Ultimately we want $a_{lm}(\bar{x}, \eta)$ so it's really

$$a_{lm}(\bar{x}, \eta) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\bar{x}} \int d\Omega Y_{lm}^*(\hat{p}) \Theta(\bar{k}, \hat{p}, \eta)$$

Same as density perturbations, we cannot make predictions for a_{lm} , just about their distribution (related to inflationary fluctuations)

$$\langle a_{lm} \rangle = 0 \quad \langle a_{lm} a_{l'm'}^* \rangle = \delta_{ll'} \delta_{mm'} \left[\frac{2}{l} \right] \rightarrow \text{what we measure}$$

~~For each l , or~~ For a given l , each a_{lm} has the same variance ($-l \leq m \leq l$)
So measuring the $(2l+1)$ a_{lm} 's for each l means sampling the distribution.
There is a fundamental limitation to how well we can measure the ζ_l 's

⇓
COSMIC VARIANCE (most pronounced at low l)

$$\left(\frac{\Delta \zeta_l}{\zeta_l} \right)_{\text{cov}} \sim \text{inverse of square root of number of samples} = \sqrt{\frac{2}{2l+1}}$$

But now we want to relate ζ_l to $\Theta_l(\bar{k}, \eta)$

Implicitly everything depends on z from now on

$a_{\ell m}(\vec{x}, \eta) = \int d^3k$

$a_{\ell m} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int d\Omega Y_{\ell m}^*(\hat{p}) \Theta(\vec{k}, \hat{p}, \eta)$ $\left\{ \langle a_{\ell m} a_{\ell' m'}^* \rangle \sim \langle \Theta(\vec{k}, \hat{p}) \Theta^*(\vec{k}', \hat{p}') \rangle \right.$

$\langle a_{\ell m} a_{\ell' m'}^* \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$

Separate these phenomena $\left\{ \begin{array}{l} \text{inflation sets} \\ \text{initial amplitude} \\ \text{and phase} \end{array} \right.$ (1)

evolution turns perturbations into anisotropies (2)

$\Theta = \delta \frac{\Theta}{\delta}$
 no \hat{p} dependence

related to transfer function
 no dependence on initial amplitude

removed from the averaging of the distribution

$\Rightarrow \langle \Theta(\vec{k}, \hat{p}) \Theta(\vec{k}', \hat{p}') \rangle = \langle \delta(\vec{k}) \delta^*(\vec{k}') \rangle \frac{\Theta(\vec{k}, \hat{p})}{\delta(\vec{k})} \frac{\Theta^*(\vec{k}', \hat{p}')}{\delta(\vec{k}')} =$

$= (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P(k) \frac{\Theta(k, \hat{k} \cdot \hat{p})}{\delta(k)} \frac{\Theta^*(k, \hat{k}' \cdot \hat{p}')}{\delta^*(k)}$ $\left\{ \begin{array}{l} \text{only depends on} \\ k, \hat{k} \cdot \hat{p}; \text{ two modes} \end{array} \right.$
 with these quantities fixed all other variables

Squaring $a_{\ell m}$ we see

$C_\ell = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int d\Omega (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P(k) Y_{\ell m}^*(\hat{p}) \frac{\Theta(k, \hat{k} \cdot \hat{p})}{\delta(k)} \int \frac{d^3k'}{(2\pi)^3} e^{i\vec{k}'\cdot\vec{x}} \int d\Omega' Y_{\ell m}(\hat{p}') \frac{\Theta^*(k', \hat{k}' \cdot \hat{p}')}{\delta^*(k')}$
 $= \int \frac{d^3k}{(2\pi)^3} P(k) \int d\Omega Y_{\ell m}^*(\hat{p}) \frac{\Theta(k, \hat{k} \cdot \hat{p})}{\delta(k)} \int d\Omega' Y_{\ell m}(\hat{p}') \frac{\Theta^*(k, \hat{k}' \cdot \hat{p}')}{\delta^*(k)}$

use $\delta^3(\vec{k} - \vec{k}')$

Now expand $\Theta(k, \hat{k} \cdot \hat{p})$ and $\Theta(k, \hat{k}' \cdot \hat{p}')$ using

$\Theta(k, \hat{k} \cdot \hat{p}) = \sum_{\ell} (-i)^\ell (2\ell+1) P_\ell(\hat{k} \cdot \hat{p}) \Theta_\ell(k)$

$$\Rightarrow C_l = \int \frac{d^3k}{(2\pi)^3} P(k) \sum_{l'l''} (-i)^{l'} (-i)^{l''} (2l'+1)(2l''+1) \frac{\Theta_{l'}(k) \Theta_{l''}^*(k)}{|S(k)|^2}$$

$$\int d\Omega P_{l'}(\hat{k} \cdot \hat{p}) Y_{lm}^*(\hat{p}) \int d\Omega' P_{l''}(\hat{k} \cdot \hat{p}') Y_{lm}(\hat{p}') =$$

$\int_{l'l''} \frac{4\pi}{2l'+1} Y_{lm}^*$ $\int_{l'l''} \frac{4\pi}{2l''+1} Y_{lm}$

$$= \frac{2}{8\pi^3} \int dk k^2 P(k) \left| \frac{\Theta_l(k)}{S(k)} \right|^2 \frac{(2l+1)^2}{(2l+1)^2} \int d\Omega_k |Y_{lm}|^2$$

= 1

$$\Rightarrow C_l = \frac{2}{\pi} \int_0^\infty dk k^2 P(k) \left| \frac{\Theta_l(k)}{S(k)} \right|^2$$

Knowing $\Theta_l(k, \eta_0) [\Theta_0(k, \eta_*) , \Theta_1(k, \eta_*) , \Psi(k, \eta_*)]$ we can plot C_l

Variance of $a_{lm} \sim$ integral over Fourier modes of variance of $\Theta_l(k)$

Anisotropy spectrum today

Start from

$$\begin{aligned} \Theta_l(k, \eta_0) \approx & [\Theta_0(k, \eta_*) + \Psi(k, \eta_*)] \mathcal{J}_l[k(\eta_0 - \eta_*)] \\ & + 3\Theta_1(k, \eta_*) \left[\mathcal{J}_{l-1}[k(\eta_0 - \eta_*)] - \frac{(l+1)\mathcal{J}_l[k(\eta_0 - \eta_*)]}{k(\eta_0 - \eta_*)} \right] \\ & + \int_0^{\eta_0} d\eta e^{-\tau} [\dot{\Psi}(k, \eta) - \dot{\Phi}(k, \eta)] \mathcal{J}_l[k(\eta_0 - \eta)] \end{aligned}$$

Large-angle anisotropies: not affected by causal microphysics, at recombination only monopole contributes

Large-angle anisotropy:

$$\Theta_0(\eta_*) + \Psi(\eta_*) \simeq \frac{\Psi(\eta_*)}{3} \simeq$$

recombination for energy
after matter-radiation equality
that we can use (now)

$$\simeq \frac{1}{3D_1(a=1)} \Psi(\eta_0) = - \frac{1}{3D_1(a=1)} \Phi(\eta_0)$$

[$\Phi \simeq \Psi$ at late times]

Recall from Poisson equation (large- k , no radiation)

$$\delta(\bar{k}, a) \simeq \frac{k^2 \Phi a}{\frac{3}{2} \Omega_m H_0^2} \Rightarrow \Theta_0(\eta_*) + \Psi(\eta_*) \simeq - \frac{\Phi(\eta_0)}{3D_1(a=1)} \simeq - \frac{\Omega_m H_0^2 \delta(\eta_0)}{2k^2 D_1(a=1)}$$

Then we can get the low- ℓ C_ℓ

$$C_\ell = \frac{2}{\pi} \int_0^\infty dk k^2 P(k) \left| \frac{\Theta_\ell(k)}{\delta(k)} \right|^2 \simeq \frac{\Omega_m^2 H_0^4}{2\pi D_1^2(a=1)} \int_0^\infty \frac{dk}{k^2} \mathcal{J}_\ell^2 [k(\eta_0 - \eta_*)] P(k)$$

$\rightarrow C_\ell^{SW}$: Sachs-Wolfe effect

$$P(k, a) \simeq 2\pi^2 \delta_H^2 \frac{k^n}{H_0^{n+3}} \underbrace{\mathcal{T}^2(k)}_{\simeq 1} \left(\frac{D_1(a)}{D_1(a=1)} \right)^2$$

large scales

$$\Rightarrow C_\ell^{SW} \simeq \pi H_0^{1-n} \left(\frac{\Omega_m}{D_1(a=1)} \right)^2 \delta_H^2 \int_0^\infty \frac{dk}{k^{2-n}} \mathcal{J}_\ell^2 [k(\eta_0 - \eta_*)]$$

$\eta_0 \ll \eta_*$
 $x = k\eta_0$
 $dk = \frac{1}{\eta_0} dx$

$$\simeq \frac{1}{2^{n-4}} \pi (\eta_0 H_0)^{1-n} \left(\frac{\Omega_m}{D_1(a=1)} \right)^2 \delta_H^2 \int_0^\infty \frac{dx}{x^{2-n}} \mathcal{J}_\ell^2(x)$$

$$= \frac{\Gamma(\ell + \frac{n}{2} - \frac{1}{2}) \Gamma(3-n)}{\Gamma(\ell + \frac{5}{2} - \frac{n}{2}) \Gamma^2(2 - \frac{n}{2})} \xrightarrow{n=1} \frac{1}{2(\ell+1)} \frac{4}{\pi}$$

$$\Rightarrow C_\ell^{SW} \simeq \frac{\pi}{2\ell(\ell+1)} \left(\frac{\Omega_m}{D_1(a=1)} \right)^2 \delta_H^2$$

$$\Rightarrow \boxed{l(l+1)C_l^{sw} = \text{const}}$$

↳ typically we plot $l(l+1)$, to see the "Sachs-Wolfe plateau"

The plateau is actually not completely flat because of:

- Θ_1 contributes

- $\dot{\Phi}, \dot{\Psi}$ non-zero when DE enters, late integrated Sachs-Wolfe effect

If $n \neq 1$, $C_l^{sw} \sim l^{n-1}$ [as $\int \frac{dx}{x^{2n}} J_2^2(x) \sim \int \frac{dl}{l^{2n}}$ peaks at $x=l$]

So already rough measurement of Sachs-Wolfe plateau excludes large deviations from $n_s \sim 1$

Small scales

Depend on monopole, dipole, and ISW effect

- Monopole: gives most of final anisotropy spectrum structure. Note:

- * monopole "zeros" not zeros but troughs due to the fact that not just one mode contributes to anisotropy at a given angular scale

- * Peak position slightly wrong compared to $l=k_0$, rather slightly smaller l ! $J_2(x)$ vs l peaks slightly below x ($\approx 0.9x$)

- * $l_p \approx 0.75 \pi \frac{l_0}{R_s}$

- Dipole: smaller than monopole, out of phase

- * raises overall anisotropy, fills in the troughs precisely because it is out of phase (lowers peak-to-trough height ratio)

- * Θ_0 and Θ_1 add incoherently (so dipole contributes less than what one would think)

• ISW effect:

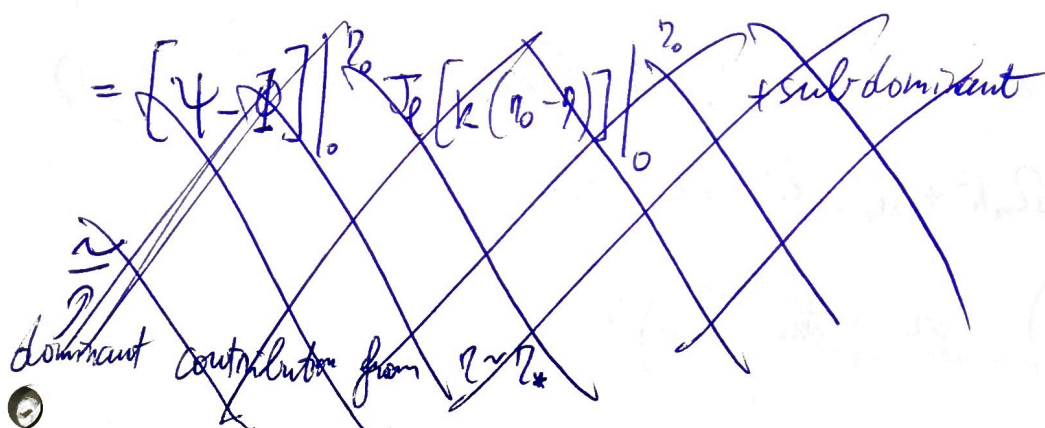
* important if potential changes after recombination

* residual radiation at recombination since matter-radiation transition is gradual

* assume potential changes at ~~some~~ time τ_c , affects all scales $k\tau_c > 1$, Bessel function peaks at $l \sim k(\tau_0 - \tau_c)$ so all scales $l > \frac{\tau_0 - \tau_c}{\tau}$ affected

* Early ISW effect adds coherently with monopole

$$\Theta_l^{eISW}(k, \tau_0) = \int_0^{\tau_0} d\tau e^{-\tau} [\dot{\Psi}(k, \tau) - \dot{\Phi}(k, \tau)] J_l[k(\tau_0 - \tau)]$$



picks up most important contributions at $\tau \sim \tau_*$

$$\Rightarrow \Theta_l^{eISW}(k, \tau_0) \approx [\Psi(k, \tau_0) - \Psi(k, \tau_*) - \Phi(k, \tau_0) + \Phi(k, \tau_*)] J_l[k(\tau_0 - \tau_*)]$$

Recall monopole

$$[\Theta_0(k, \tau_*) + \Psi(k, \tau_*)] J_l[k(\tau_0 - \tau_*)]$$

↳ adds in phase with the monopole, same Bessel function

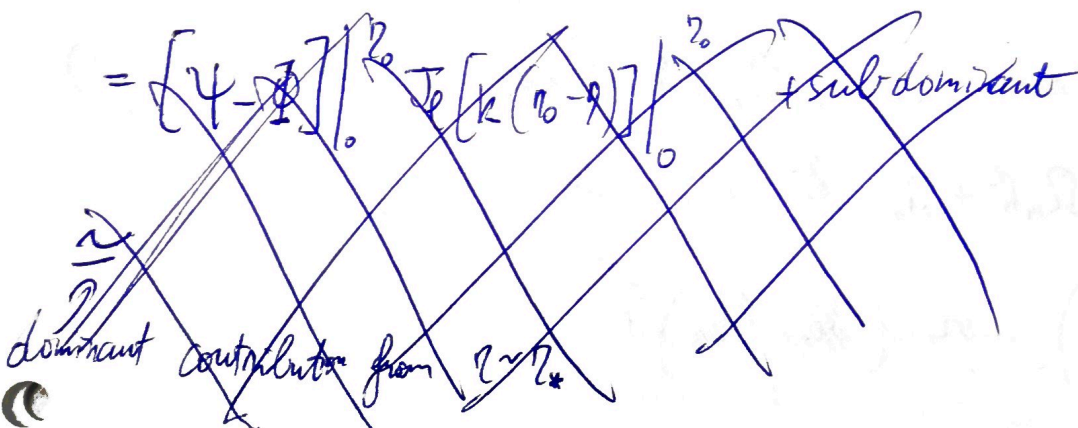
↳ Boosts first peak

ISW effect:

- * important if potential changes after recombination
- * residual radiation at recombination since matter-radiation transition is gradual
- * assume potential changes at ~~some~~ time η_c , affects all scales $k\eta_c > 1$, Bessel function peaks at $l \sim k(\eta_0 - \eta_c)$ so all scales $l > \frac{\eta_0 - \eta_c}{\eta_c}$ affected
- * Early ISW effect adds coherently with monopole

$$\Theta_{\ell}^{ISW}(k, \eta_0) = \int_0^{\eta_0} d\eta e^{-\tau} [\dot{\Psi}(k, \eta) - \ddot{\Phi}(k, \eta)] J_{\ell}[k(\eta_0 - \eta)]$$

$$= \left[\dot{\Psi} - \ddot{\Phi} \right] \Big|_0^{\eta_0} J_{\ell}[k(\eta_0 - \eta)] \Big|_0^{\eta_0} + \text{subdominant}$$



picks up most important contributions at $\eta \sim \eta_*$

$$\Rightarrow \Theta_{\ell}^{ISW}(k, \eta_0) \approx [\Psi(k, \eta_0) - \Psi(k, \eta_*) - \Phi(k, \eta_0) + \Phi(k, \eta_*)] J_{\ell}[k(\eta_0 - \eta_*)]$$

Recall monopole

$$[\Theta_0(k, \eta_*) + \Psi(k, \eta_*)] J_{\ell}[k(\eta_0 - \eta_*)]$$

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Cosmological parameters

Possible parameters:

- $\Omega_k = 1 - \Omega_m - \Omega_\Lambda$
- $C_{l_0} \rightarrow A_s$
- n_s
- r
- $\ln \tau(z_{re})$
- $\Omega_b h^2 \rightarrow \omega_b$
- $\Omega_m h^2 \rightarrow \omega_m$
- Ω_Λ

6 NCDM parameters:

- ω_b
- ω_c
- $\Omega_s \leftrightarrow H_0$
- A_s
- n_s
- τ

Note: in a flat universe $\Omega_m + \Omega_\Lambda \approx 1$ (neglect radiation today!)

$$\Omega_m + \Omega_\Lambda = 1 \Rightarrow \Omega_m h^2 + \Omega_\Lambda = h^2 \Rightarrow \frac{\Omega_m h^2}{h^2} + \Omega_\Lambda = 1$$

Vary $(\Omega_m h^2, \Omega_\Lambda)$ or (h, Ω_Λ) ?

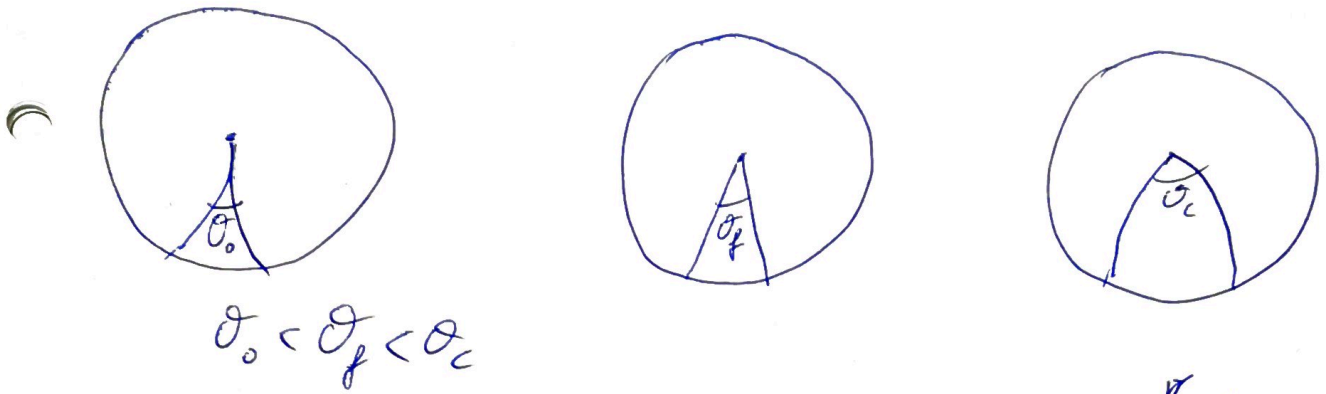
↓
raise Ω_Λ fixing $\Omega_m h^2$
req fixed

↓
raise Ω_Λ but lower Ω_m
to keep $\Omega_m h^2 + \Omega_\Lambda = h^2$
while fixing h , so
req is lower ($\Omega_m h^2$ lower)

Curvature

Open Universe \rightarrow smaller angles \rightarrow peaks to ^{larger} smaller l

Closed Universe \rightarrow larger angles \rightarrow peaks to ^{larger} smaller l



Peak position depends on $l_p \propto \frac{r_0}{r_s} \propto \left[\int \frac{dz}{H(z)} \right]$

$$\left[l = k r_0 \rightarrow l_p = k_p r_0 = \frac{n \pi r_0}{r_s} \right]$$

~~$H(z) = H_0 \sqrt{\Omega_m (1+z)^3 + \Omega_b + \Omega_r (1+z)^4}$~~ ~~closed $\Omega_m < 1$~~

Angular diameter distance to last-scattering larger in an open universe

$$D_A \propto (1 - \Omega_k)^{-0.45}$$

Degenerate parameters

- C_{10} moves spectrum up and down
- n_s if < 1 suppresses small relative to large scales

$$\frac{C_l(n_s)}{C_l(n_s=1)} \approx \left(\frac{l}{l_{\text{pivot}}} \right)^{n_s - 1} \quad P(k) \propto k^{n_s}$$

- τ $\underbrace{T(1+\Theta)}_{\text{escape}} e^{-\tau} + \underbrace{T(1-e^{-\tau})}_{\text{from reionization}} = T(1+\Theta e^{-\tau}) \Rightarrow \Theta e^{-\tau}$

suppresses all scales within horizon at reionization

- r Enhances anisotropies for $l < 100$

Distinct Imprints

- $\Omega_b h^2$: shifts peaks due to shift in r_s , affects odd-to-even peak height changes diffusion scale
- Ω_n : late isre effect. enhances ~~large~~ large-scale anisotropies
- $\Omega_m h^2$: if lowered, enhances early isre effect