BEYOND EQUILIBRIUM: INTRODUCTION TO THE BOLTZMANN EQUATIONS

(~Dodelson chapter 3)

Early Universe → hot and dense → frequent interactions

(almost always) equilibrium

Things get interesting out of equilibrium. 3 case studies:

1) $T \sim \text{GeV}$? Production of DM ("freeze-out")
2) $T \sim \text{MeV}$ BBN ("fusion" of light elements)
3) $T \sim 0.1\,\text{eV}$ CMB ("recombination" of neutral hydrogen)

non-equilibrium physics: Boltzmann equations!

We will first study it under certain approximations (integrated form)

Boltzmann equations: rate of change in abundance of particle

=

rate for production − rate for elimination

Boltzmann equation for annihilation

$$1 + 2 \longleftrightarrow 3 + 4$$

Follow particle 1

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4}$$
$$\times (2\pi)^4 \delta^3(p_1 + p_2 - p_3 - p_4)\, \delta(E_1 + E_2 - E_3 - E_4)\, |M|^2$$
$$\times \left\{ f_3 f_4 [1 \pm f_1][1 \pm f_2] - f_1 f_2 [1 \pm f_3][1 \pm f_4] \right\}$$

If no interactions:

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = 0 \longrightarrow n_1 \propto a^{-3}$$

Forget $1 \pm f$ terms for the moment (Bose enhancement / Pauli blocking)

$$\frac{d(n_1 a^3)}{dt} \propto \underbrace{f_3 f_4(p)}_{\text{production rate}} - \underbrace{f_1 f_2(p)}_{\text{loss term}}$$

$\delta^3(p_1 + p_2 - p_3 - p_4)$: momentum conservation

$\delta(E_1 + E_2 - E_3 - E_4)$: energy conservation $\quad E = \sqrt{p^2 + m^2}$

$(2\pi)^n$ from discrete $\rightarrow$ continuous δ

$|M|$: matrix element (amplitude) for fundamental process $\rightarrow$ from QFT (Feynman rules)

$\int \frac{1}{(2\pi)^3} = \int \frac{1}{(2\pi\hbar)^3} \rightarrow$ integral over unit of phase space
need to sum/integrate over all momenta

why $\frac{1}{2E}$? $\quad \int d^4p\, \delta(E^2 - p^2 - m^2) = \int d^3p \int dE\, \delta(E^2 - p^2 - m^2)$ $*$

$$\left(\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \longrightarrow \delta(E^2 - p^2 - m^2) = \frac{\delta(E - \sqrt{p^2 + m^2})}{2E} \right)$$

$\uparrow$ only $E = +\sqrt{p^2 + m^2}$

$$* = \int d^3p \int \frac{dE\, \delta(E - \sqrt{p^2 + m^2})}{2E} = \int \frac{d^3p}{2\sqrt{p^2 + m^2}} = \int \frac{d^3p}{2E(p)}$$

We have equation for n_1, but also need for n_2, n_3, n_4

↳ Coupled integro-differential equations!

↳ approximations!

Approximations:

- Kinetic equilibrium (enforced by scattering) → scattering so rapid that species distributions are of the BE/FD form

$$f = \left[e^{\frac{E-\mu}{T}} \pm 1\right]^{-1} \quad \text{only 1 parameter } \mu$$

- chemical equilibrium is <u>NOT</u> guaranteed! (not always)

- $T \ll E-\mu \implies f \approx \left(e^{\frac{E-\mu}{T}}\right)^{-1} = e^{\frac{\mu}{T}} e^{-\frac{E}{T}}$

↳ can neglect Pauli blocking / Bose enhancement $(1 \pm f) \sim 1$

Look again at last line of Boltzmann equation

$$f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4) \approx f_3 f_4 - f_1 f_2 =$$

$$= e^{\frac{\mu_3}{T}} e^{-\frac{E_3}{T}} e^{\frac{\mu_4}{T}} e^{-\frac{E_4}{T}} - e^{\frac{\mu_1}{T}} e^{-\frac{E_1}{T}} e^{\frac{\mu_2}{T}} e^{-\frac{E_2}{T}} =$$

$$= e^{-\frac{(E_3+E_4)}{T}} e^{\frac{(\mu_3+\mu_4)}{T}} - e^{-\frac{(E_1+E_2)}{T}} e^{\frac{(\mu_1+\mu_2)}{T}} =$$

↳ $E_1 + E_2 = E_3 + E_4$

$$= e^{-\frac{(E_1+E_2)}{T}} \left[e^{\frac{(\mu_3+\mu_4)}{T}} - e^{\frac{(\mu_1+\mu_2)}{T}} \right]$$

Reduced Boltzmann equation to 1 (N) equation(s) for μ_i

Integrate to get number density

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f(p) = g_i e^{\frac{\mu_i}{T}} \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E_i}{T}}$$

$$n_i^{(0)} : \text{at equilibrium}_{\mu=0} \Rightarrow g_i \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E_i}{T}} = \begin{cases} g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-\frac{m_i}{T}} & m_i \gg T \\ g_i \frac{T^3}{\pi^2} & m_i \ll T \end{cases}$$

↳ $e^{\frac{\mu_i}{T}} = \frac{n_i}{n_i^{(0)}}$ (pretty much by definition)

$$e^{-\frac{(E_1+E_2)}{T}} \left\{ e^{\frac{(\mu_3+\mu_4)}{T}} - e^{\frac{(\mu_1+\mu_2)}{T}} \right\} = e^{-\frac{(E_1+E_2)}{T}} \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\}$$

$$f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4) \approx e^{-\frac{(E_1+E_2)}{T}} \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\}$$

Define thermally-averaged cross-section

$$\langle \sigma v \rangle \equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \cdots \int \frac{d^3 p_4}{(2\pi)^3 2E_4} \, e^{-\frac{(E_1+E_2)}{T}} (2\pi)^4 \delta^3 (p_1+p_2-p_3-p_4)\, \delta(E_1+E_2-E_3-E_4)$$
$$\times |\mathcal{M}|^2$$

$\Rightarrow$ Boltzmann equation simplifies to

$$\boxed{\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right)}$$

$\hookrightarrow$ Integrated Boltzmann equation

LHS $\sim \frac{n_1}{t} \sim n_1 H$

RHS $\sim n_1 n_2 \langle \sigma v \rangle \sim n_1 \Gamma$

If $\Gamma \gg H$ $\longrightarrow$ RHS $\gg$ LHS $\longrightarrow (\ldots) = 0$, $n_1 \propto a^{-3}$

equilibrium

$$\boxed{\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}}$$

chemical equilibrium / Nuclear statistical equilibrium $\leftarrow$

<u>Saha equation</u>

$1+2 \Leftrightarrow 3+4$

Applications:

		1	2	3	4
a)	DM production	X	X	l	l
b)	BBN	n	ν_e/e^+	p	$e^-/\bar{\nu}_e$
c)	Recombination	e	p	H	γ