Now let's apply the Boltzmann equation ~~(or rather its Saha equation approximation)~~

Big Bang Nucleosynthesis

$T \sim 1\,MeV$ (typical nuclear binding energies)

Plasma consists of:

- relativistic particles in equilibrium

  γ, e, e^+ $\quad e^+e^- \leftrightarrow \gamma\gamma$

- decoupled relativistic neutrinos

  $\nu e \leftrightarrow \nu e$ decouples at $T \sim MeV$

- non-relativistic baryons

  p, n

  They have not annihilated only thanks to a primordial asymmetry $\frac{n_b - n_{\bar{b}}}{s} \sim 10^{-10}$

At $T \sim MeV$ $\quad \eta_b \equiv \frac{n_b}{n_\gamma} \sim 10^{-10}$ baryon-to-photon ratio

Many more γ than baryons!

BBN tells us how p, n end up: not Fe, but lighter nuclei, thanks to nuclear reactions

Simplifications:

- Track H and He <u>only</u> (and their isotopes): ^1H, ^2H, ^3H, ^3He, ^4He
- $T \gtrsim 0{,}1\,MeV$: only free n, p

  $\hookrightarrow$ solve for $\frac{n}{p}$ ratio, use this as input for synthesis of ^4He

Are these good approximations? Take e.g. Deuterium production

$$n + p \longleftrightarrow D + \gamma$$

Saha equation (considering γ in equilibrium $\to n_\gamma = n_\gamma^{(0)}$)

$$\frac{n_D}{n_n n_p} = \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}} \qquad n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-\frac{m_i}{T}}$$

since $m_i \gg T$

$g_i = 3\,(D),\ 2\,(n),\ 2\,(p)$

$$\Rightarrow \frac{n_D}{n_n n_p} = \frac{3}{4}\left(\frac{2\pi m_D}{m_n m_p T}\right)^{3/2} e^{\frac{m_n + m_p - m_D}{T}}$$

$B_D \equiv m_n + m_p - m_D \simeq 2.22$ MeV binding energy of Deuterium

$$\Rightarrow \frac{n_D}{n_n n_p} \approx \frac{3}{4}\left(\frac{4\pi}{m_p T}\right)^{3/2} e^{\frac{B_D}{T}}$$

$m_D \simeq 2m_n \simeq 2m_p$ in prefactor

~~And~~ $n_n, n_p \sim n_b = n_\gamma \eta_b \sim \eta_b T^3 \Rightarrow \frac{n_D}{n_n n_p} \sim \frac{n_D}{n_b \eta_b T^3}$

$$\Rightarrow \underbrace{\frac{n_D}{n_b}}_{\text{small!}} \sim \underbrace{\eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{\frac{B_D}{T}}}_{\text{small, dominated by pre-factor unless } e^{\frac{B_D}{T}} \text{ large!}}$$

~~As~~ Until $T \ll B_D$, both LHS and RHS tiny $\to \frac{n_D}{\eta_b} \ll 1$

↳ deuterium cannot be produced, only free p and n

At $T \sim$ MeV BBN starts, produces elements up to $\sim {}^4He$

Strategy: determine $\frac{n_n}{n_p}$, use it as input for subsequent reactions

Neutron fraction $\quad X_n \equiv \frac{n_n}{n_n + n_p} \xrightarrow[\text{equilibrium}]{} X_{n,eq} \equiv \frac{1}{1 + \frac{n_p^{(0)}}{n_n^{(0)}}}$

Essentially we want to track X_n down to $\sim$ meV

Boltzmann equation:

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right)$$

$$\underset{1}{n} + \underset{2}{l} \longleftrightarrow \underset{3}{p} + \underset{4}{l}$$

$\underbrace{l = \nu_e, e, \bar{\nu}, e^+}_{\text{in equilibrium! } n_l = n_l^{(0)}}$

$$\frac{1}{a^3}\frac{d(n_n a^3)}{dt} = n_l^{(0)} n_n^{(0)} \langle \sigma v \rangle \left(\frac{n_p \cancel{n_l}}{n_p^{(0)} \cancel{n_l^{(0)}}} - \frac{n_n \cancel{n_l}}{n_n^{(0)} \cancel{n_l^{(0)}}} \right) =$$

$$= n_l^{(0)} \langle \sigma v \rangle \left(\frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n \right)$$

$$\frac{n_n^{(0)}}{n_p^{(0)}} = \left(\frac{m_n}{m_p}\right) e^{-\frac{(m_n - m_p)}{T}} \equiv \underbrace{\left(\frac{m_n}{m_p}\right)}_{\sim 1} e^{-\frac{Q}{T}} \approx e^{-\frac{Q}{T}}$$

$Q \equiv m_n - m_p \simeq 1.3$ MeV

$n_l^{(0)} \langle \sigma v \rangle \equiv \lambda_{np}$ rate for neutron-proton conversion

$n_n = (n_n + n_p) X_n$ ~~$\rightarrow a^{-3} \frac{d(n_n a^3)}{dt} = a^{-3} \frac{d[(n_n + n_p) X_n a^3]}{dt}$~~

$$\rightarrow a^{-3}\frac{d(n_n a^3)}{dt} = a^{-3}\frac{d[(n_n + n_p) a^3 X_n]}{dt} \underset{\uparrow}{=} \cancel{a^{-3}} (n_n + n_p) \cancel{a^3} \frac{dX_n}{dt} = (n_n + n_p)\frac{dX_n}{dt}$$

$(n_n + n_p) a^3 \sim n_b a^3 = \text{const}$

$$(n_n+n_p)\frac{dX_n}{dt} = \lambda_{np}\left(\frac{n_p\, n_n(0)}{n_p(0)} - n_n\right) = \lambda_{np}\left(n_p e^{-\frac{Q}{T}} - n_n\right)$$

$$\Rightarrow \boxed{\frac{dX_n}{dt} = \lambda_{np}\left(\frac{n_p}{n_n+n_p}e^{-\frac{Q}{T}} - \frac{n_n}{n_n+n_p}\right) = \lambda_{np}\left[(1-X_n)e^{-\frac{Q}{T}} - X_n\right]}$$

$$\boxed{\frac{dX_n}{dt} = \lambda_{np}\left[(1-X_n)e^{-\frac{Q}{T}} - X_n\right]}$$

note: $T, \lambda_{np} = T(t), \lambda_{np}(t)$!

New variable $x \equiv \frac{Q}{T}$

$$\cancel{\frac{dX_n}{dt} = \frac{dX_n}{dx}\frac{dx}{dt} = -x\frac{\dot T}{T}\frac{dX_n}{dx}\frac{dx}{dt}}$$

$\frac{\dot T}{T} = -H$ since $T \propto a^{-1}$

$$\frac{dX_n}{dt} = \frac{dX_n}{dx}\frac{dx}{dt} = -\frac{dX_n}{dt}\, x\frac{\dot T}{T} = xH\frac{dX_n}{dx}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{Q}{T}\right) = -\frac{Q}{T^2}\dot T = -\frac{Q}{T}\frac{\dot T}{T} = -x\frac{\dot T}{T}$$

$$\frac{dX_n}{dt} = xH\frac{dX_n}{dx} \qquad H = \sqrt{\frac{8\pi G\rho}{3}} \qquad \rho = \frac{\pi^2}{30}T^4\left[\sum_{\text{bosons}} g_i + \frac{7}{8}\sum_{\text{fermions}} g_f\right] \equiv$$

$$\equiv g_*(T)\frac{\pi^2}{30}T^4$$

at $T\sim\text{MeV}$ relativistic particles : γ $(g_\gamma=2)$, ν $(g_\nu=6)$, $e^\pm$ $(g_{e^+}=g_{e^-}=2)$

$\Rightarrow g_* \simeq 10.75$ constant in $T\sim\text{MeV}$

$$xH\frac{dX_n}{dx} = x\frac{dX_n}{dx}\sqrt{\frac{8\pi G}{3}g_*\frac{\pi^2}{30}T^4} = xT^2\frac{dX_n}{dx}\sqrt{\frac{8\pi G}{3}g_*\frac{\pi^2}{30}} = \cancel{x}\frac{Q^2}{x^{\cancel{2}}}\sqrt{\frac{4\pi^3 G}{45}}\times\sqrt{10.75} =$$

$$= \frac{1}{x}\sqrt{\frac{4\pi^3 GQ^4}{45}}\times\sqrt{10.75} \equiv H(T=Q) = H(x=1) \simeq 1.1\ s^{-1}$$

$$\frac{dX_n}{dt} = \lambda_{np}\left[(1-X_n)\,e^{-\frac{Q}{T}} - X_n\right]$$

$$\hookrightarrow \quad \frac{H(x=1)}{x}\frac{dX_n}{dx} = \lambda_{np}\left[(1-X_n)\,e^{-\frac{Q}{T}} - X_n\right]$$

$$\hookrightarrow \quad \frac{dX_n}{dx} = \frac{x\,\lambda_{np}}{H(x=1)}\underbrace{\left[(1-X_n)\,e^{-x} - X_n\right]}$$

$$= e^{-x} - X_n e^{-x} - X_n = e^{-x} - X_n\left(1+e^{-x}\right)$$

$$\Downarrow$$

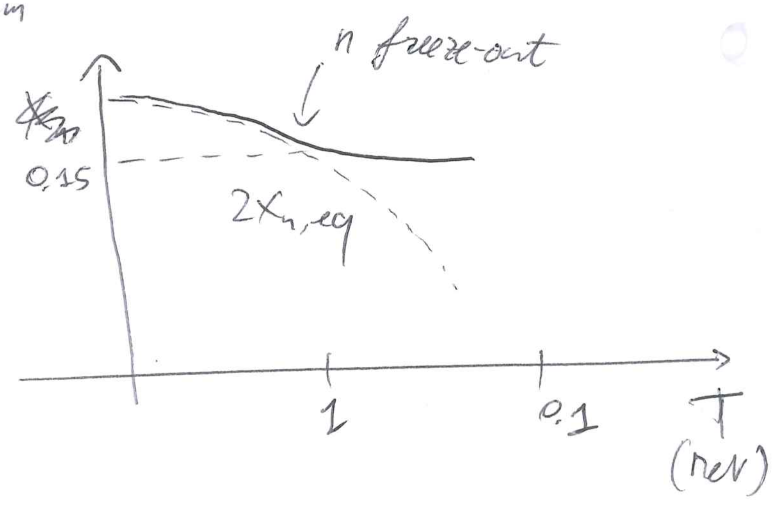
$$\boxed{\frac{dX_n}{dx} = \frac{x\,\lambda_{np}}{H(x=1)}\left[e^{-x} - X_n\left(1+e^{-x}\right)\right]}$$

$\lambda_{np} \approx \frac{255}{\tau_n x^5}\left(12+6x+x^2\right)$ $\qquad \tau_n \approx 886.7\ s$

$T \sim MeV \quad x \sim 1 \rightarrow \lambda_{np} \sim 5.5\,s^{-1}, \quad H \sim 1.1\,s^{-1}$

$\lambda_{np} > H \Rightarrow$ still equilibrium

Integrate $\frac{dX_n}{dx}$ numerically



For $T \lesssim 0.5$ MeV, X_n freezes out at $X_n \sim 0.15$

Below 0.1 MeV need to include:

- neutron decay $n \rightarrow p + e^- + \bar{\nu}_e$
- deuterium production $n + p \rightarrow D + \gamma$

Production of D, He only really starts at $T_{nuc} \sim 0.07$ MeV

Neutron fraction depleted by $e^{-\frac{t}{\tau_n}} \longrightarrow X_n(T_{nuc}) \approx 0.11$

So nucleosynthesis starts with an input neutron fraction

$$X_n \simeq 0.11$$

$$\underset{\sim 1}{\frac{n_D}{n_b}} \sim \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{\frac{B_D}{T}} \longrightarrow \underbrace{\ln(\eta_b) + \frac{3}{2}\ln\left(\frac{T_{nuc}}{m_p}\right) \sim -\frac{B_D}{T_{nuc}}}_{T_{nuc} \sim 0.07\,\mathrm{MeV}}$$

~~Be~~ B higher for ^4He than D $\longrightarrow$ ^4He production favored

So at $T \simeq T_{nuc}$ all remaining neutrons processed into ^4He

Need 2 neutrons for $^4He \rightarrow n_{^4He} = \frac{n_n}{2}$

Mass fraction

$$\cancel{X_4} \equiv Y_p \equiv \frac{4 n_{^4He}}{n_b} = 2X_n(T_{nuc}) \approx 0.22$$

$$Y_p \approx 0.2262 + 0.0135 \ln\left(\frac{\eta_b}{10^{-10}}\right)$$ Logarithmic sensitivity to Ω_b

$\hookrightarrow$ Excellent agreement with observations!

Pillar of observational cosmology

Besides ^4He, BBN produces some D (excellent indicator of $\Omega_b h^2$!) and ^7Li (disagreement theory - observations)

<u>Recombination</u>

$T \gtrsim 1\,\mathrm{eV}$ $\qquad \gamma \underset{\text{Compton scattering}}{\longleftrightarrow} e^- \underset{\text{Coulomb scattering}}{\longleftrightarrow} p^+$

All tightly coupled $\rightarrow$ no neutral Hydrogen

$B \simeq 13.6\,\mathrm{eV}$, but high η delays formation of H until $T \sim 0.25\,\mathrm{eV}$

$$e^- + p \longleftrightarrow H + \gamma$$
$$\;1 \quad\; 2 \qquad\quad 3 \quad 4$$

While in equilibrium, Saha equation is valid

$$\frac{n_e n_p}{n_H \cancel{n_\gamma}} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)} \cancel{n_\gamma^{(0)}}} \xRightarrow[n_\gamma = n_\gamma^{(0)}]{} \frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

Neutrality of the Universe $\longrightarrow$ $n_e = n_p$

Free electron fraction $\quad X_e \equiv \dfrac{n_e}{n_e + n_H} = \dfrac{n_p}{n_p + n_H} = \dfrac{n_e}{n_b}$

$$1 - X_e = \frac{n_H}{n_e + n_H} = \frac{n_H}{n_p + n_H} = \frac{n_H}{n_b}$$

$$\frac{n_e n_p}{n_H} = \frac{X_e (n_e + n_H)\, X_e (n_e + n_H)}{(1 - X_e)(n_e + n_H)} = \frac{X_e^2}{1 - X_e} (n_e + n_H)$$

B = hydrogen binding energy (ε)

$$\frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{(m_e + m_p - m_H)}{T}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\varepsilon_0}{T}}$$

$n_i^{(0)} = g_i \left(\dfrac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}$, $m_p \sim m_H$ in pre-factor

$$\rightarrow \frac{X_e^2}{1 - X_e} = \underbrace{\frac{1}{n_e + n_H} \left[\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\varepsilon_0}{T}}\right]}$$

Denominator $n_e + n_H \sim n_b = \eta_b n_\gamma \sim 10^{-9} T^3$

$T \sim \varepsilon_0 \longrightarrow RHS \sim 10^9 \left(\frac{m_e}{T}\right)^{3/2} \simeq 10^{15}$!!!

LHS = RHS only if $X_e \simeq 1$ = no ~~H~~ hydrogen!