

# BEYOND EQUILIBRIUM:

## INTRODUCTION TO THE BOLTZMANN EQUATIONS

(~Dodelson Chapter 3)

Early Universe  $\rightarrow$  hot and dense  $\rightarrow$  frequent interactions

(almost always) equilibrium  $\leftarrow$

Things get interesting out of equilibrium. 3 case studies:

- 1)  $T \sim \text{GeV?}$  Production of DM ("freeze-out")
- 2)  $T \sim \text{MeV}$  BBN ("fusion" of light elements)
- 3)  $T \sim 0.1 \text{ eV}$  CMB ("recombination" of neutral hydrogen)

non-equilibrium physics: Boltzmann equations!

We will first study it under certain approximations (integrated form)

Boltzmann equations: rate of change in abundance of particle

=

rate for production - rate for elimination

Boltzmann equation for annihilation

$$1 + 2 \leftrightarrow 3 + 4$$

Follow particle 1

$$\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int \frac{d^3 p_4}{(2\pi)^3 2E_4}$$
$$\times (2\pi)^4 \delta^3(p_1 + p_2 - p_3 - p_4) \delta(E_1 + E_2 - E_3 - E_4) |M|^2$$
$$\times \{ f_3 f_4 [1 \pm f_1][1 \pm f_2] - f_1 f_2 [1 \pm f_3][1 \pm f_4] \}$$

If no interactions:

$$\frac{1}{a^3} \frac{d(n_i a^3)}{dt} = 0 \rightarrow n_i \propto a^{-3}$$

Forget  $\pm$  terms for the moment (Bose enhancement/Pauli blocking)

$$\frac{d(n_i a^3)}{dt} \propto \underbrace{f_3 f_4(p)}_{\text{production rate}} - \underbrace{f_1 f_2(p)}_{\text{loss term}}$$

$\delta^3(p_1 + p_2 - p_3 - p_4)$ : momentum conservation

$\delta(E_1 + E_2 - E_3 - E_4)$ : energy conservation

$$E = \sqrt{p^2 + m^2}$$

$(2\pi)^4$  from discrete  $\rightarrow$  continuous  $\delta$

$|M|$ : matrix element (amplitude) for fundamental process  $\rightarrow$  from QFT (Feynman rules)

$$\int \frac{1}{(2\pi)^3} = \int \frac{1}{(2\pi h)^3} \rightarrow \text{integral over unit of phase space needed to sum/integrate over all momenta}$$

why  $\frac{1}{2E}$ ?  $\int d^4 p \delta(E^2 - p^2 - m^2) = \int d^3 p \int dE \delta(E^2 - p^2 - m^2) *$

$$\left( \delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \rightarrow \delta(E^2 - p^2 - m^2) = \frac{\delta(E^2 - \sqrt{p^2 + m^2})}{2E} \right)$$

$\uparrow$  only  $E = + \sqrt{p^2 + m^2}$

$$* = \int d^3 p \int \frac{dE}{2E} \delta(E - \sqrt{p^2 + m^2}) = \int \frac{d^3 p}{2\sqrt{p^2 + m^2}} = \int \frac{d^3 p}{2E(p)}$$

We have equation for  $n_1$ , but also need for  $n_2, n_3, n_4$

$\hookrightarrow$  coupled integro-differential equations!

$\hookrightarrow$  approximations!

## Approximations:

- Kinetic equilibrium (enforced by scattering)  $\rightarrow$  scattering so rapid that species distributions are of the BE/FD form

$$f = \left[ e^{\frac{E-\mu}{T}} \pm 1 \right]^{-1} \quad \text{only 1 parameter } \mu$$

- chemical equilibrium is NOT guaranteed! (not always)

$$T \ll E - \mu \Rightarrow f \approx \left( e^{\frac{E-\mu}{T}} \right)^{-1} = e^{\frac{\mu}{T}} e^{-\frac{E}{T}}$$

$\hookrightarrow$  can neglect Pauli blocking / Bose enhancement  
 $(1 \pm f) \approx 1$

Look again at last line of Boltzmann equation

$$f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4) \approx f_3 f_4 - f_1 f_2 =$$

$$= e^{\frac{\mu_3}{T}} e^{-\frac{E_3}{T}} e^{\frac{\mu_4}{T}} e^{-\frac{E_4}{T}} - e^{\frac{\mu_1}{T}} e^{-\frac{E_1}{T}} e^{\frac{\mu_2}{T}} e^{-\frac{E_2}{T}} =$$

$$= e^{-\frac{(E_3+E_4)}{T}} e^{\frac{(\mu_3+\mu_4)}{T}} - e^{-\frac{(E_1+E_2)}{T}} e^{\frac{(\mu_1+\mu_2)}{T}} =$$

$$= e^{-\frac{(E_1+E_2)}{T}} \left[ e^{\frac{(\mu_3+\mu_4)}{T}} - e^{\frac{(\mu_1+\mu_2)}{T}} \right]$$

$$\uparrow E_1 + E_2 = E_3 + E_4$$

Reduced Boltzmann equation to 1(N) equation(s) for  $\mu_i$

Integrate to get number density

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f(p) = g_i e^{\frac{\mu_i}{T}} \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E_i}{T}}$$

$$n_i^{(0)} : \text{at equilibrium } \mu \Rightarrow g_i \int \frac{d^3p}{(2\pi)^3} e^{-\frac{E_i}{T}} = \begin{cases} g_i \left( \frac{m_i T}{2\pi} \right)^{3/2} e^{-\frac{\mu_i}{T}} & m_i \gg T \\ g_i \frac{T^3}{\pi^2} & m_i \ll T \end{cases}$$

$$\hookrightarrow e^{\frac{\mu_i}{T}} = \frac{n_i}{n_i^{(0)}} \quad (\text{pretty much by definition})$$

$$e^{-\frac{(E_1+E_2)}{T}} \left\{ e^{\frac{(m_3+m_4)}{T}} - e^{\frac{(m_1+m_2)}{T}} \right\} = e^{-\frac{(E_1+E_2)}{T}} \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\}$$

$$f_3 f_4 (1 \pm f_1)(1 \pm f_2) - f_1 f_2 (1 \pm f_3)(1 \pm f_4) \approx e^{-\frac{(E_1+E_2)}{T}} \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\}$$

Define thermally-averaged cross-section

$$\langle \sigma v \rangle \equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \dots \int \frac{d^3 p_4}{(2\pi)^3 2E_4} e^{-\frac{(E_1+E_2)}{T}} (2\pi)^4 \delta^3(p_1+p_2-p_3-p_4) \delta(E_1+E_2-E_3-E_4) \times |M|^2$$

⇒ Boltzmann equation simplifies to

$$\frac{1}{a^3} \frac{d(n_i a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left[ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right]$$

LHS  $\sim \frac{\dot{n}_i}{t} \sim n_i H$

RHS  $\sim n_1 n_2 \langle \sigma v \rangle \sim n_i^2$

↳ Integrated Boltzmann equation

If  $(n \gg H) \rightarrow$  RHS  $\gg$  LHS  $\rightarrow (\dots) = 0$ ,  $n_i \propto a^{-3}$

Chemical equilibrium /  
Nuclear statistical equilibrium  
Saha equation

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}$$

$$1+2 \leftrightarrow 3+4$$

Applications:

a) DM production

b) BBN

c) Recombination

1	2	3	4
X	X	l	l
n	$\nu_e/\bar{\nu}_e$	p	$e^{-1/2}$
e	p	H	$\gamma$