

High γ prevents H formation even when initially energetically favorable $T \sim \epsilon_0$. Need $T \ll \epsilon_0$ before recombination can proceed

X_e drops \rightarrow process out of equilibrium, cannot use Saha equation, need Boltzmann equation

$$\frac{1}{a^3} \frac{d(n_e a^3)}{dt} = n_e^{(0)} n_p^{(0)} \langle \sigma v \rangle \left\{ \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}} - \frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} \right\} =$$

$$= n_e^{(0)} n_p^{(0)} \langle \sigma v \rangle \left[\frac{n_H}{n_H^{(0)}} - \frac{n_e^2}{n_e^{(0)} n_p^{(0)}} \right] =$$

$$= n_b \langle \sigma v \rangle \left[\frac{n_H}{n_b} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} - \frac{n_e^2}{n_b} \right]$$

$$\begin{aligned} n_e &= X_e n_b \\ n_H &= (1 - X_e) n_b \\ \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} &\approx \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\frac{\epsilon_0}{T}} \end{aligned}$$

$$= n_b \langle \sigma v \rangle \left[(1 - X_e) \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\frac{\epsilon_0}{T}} - X_e^2 n_b \right]$$

$$\frac{1}{a^3} \frac{d(n_e a^3)}{dt} = \frac{1}{a^3} \frac{d(n_b X_e a^3)}{dt} = \frac{1}{a^3} n_b a^3 \frac{dX_e}{dt} = n_b \frac{dX_e}{dt}$$

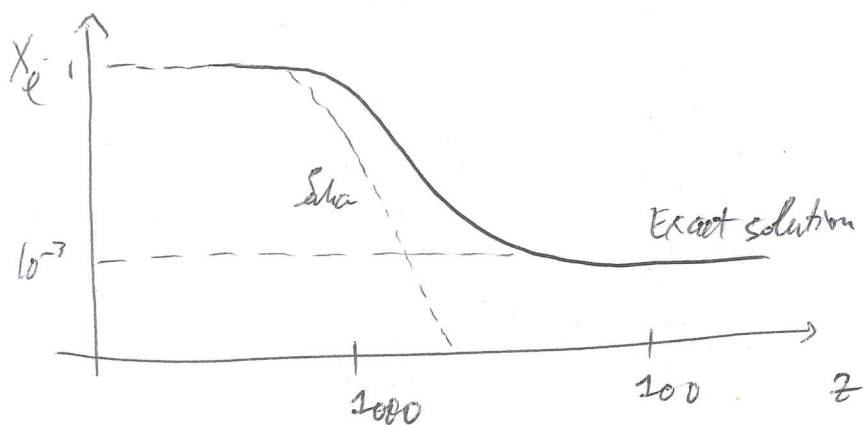
$\underbrace{\frac{1}{a^3} n_b a^3}_{n_b a^3 = \text{const}}$

$$\Rightarrow \frac{dX_e}{dt} = \langle \sigma v \rangle \left[(1 - X_e) \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\frac{\epsilon_0}{T}} - X_e^2 n_b \right] = \left[(1 - X_e) \beta - X_e^2 n_b \alpha^{(2)} \right]$$

$$\beta \equiv \langle \sigma v \rangle \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-\frac{\epsilon_0}{T}} \quad \text{ionization rate}$$

$$\alpha^{(2)} \equiv \langle \sigma v \rangle \approx 9.78 \frac{\alpha^2}{h_e^2} \left(\frac{\epsilon_0}{T} \right)^{1/2} \ln \left(\frac{\epsilon_0}{T} \right) \quad \text{recombination rate}$$

Need to solve numerically for $X_e(t)$



Saha solution falls out of equilibrium, but is good approximation

Recombination occurs when $X_e^{(S)} \sim 10^{-2}$
 ↑
 from Saha!

$z_* \sim 1000$ redshift of recombination

When $X_e \sim 10^{-2}$ photons decouple because Compton scattering becomes inefficient ($\Gamma \sim X_e n_b \sigma_T$). So (b) decoupling happens roughly during recombination, in fact the two terms are often used interchangeably.

Dark matter freeze-out (hypothetical!!!)

Weakly interacting massive particle (WIMP, $m \sim \text{GeV}$, $\sigma \sim \sigma_{\text{sc}} \sim 10^{-39} \text{ cm}^2$)

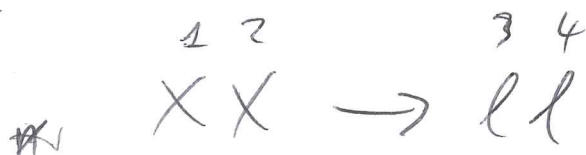
in thermal equilibrium at high temperatures ($XX \rightarrow ll$)

Freeze-out: inability of annihilations to maintain equilibrium

If this particle were kept in equilibrium indefinitely, $n \sim e^{-\frac{m}{T}} \rightarrow 0!$

Because of freeze-out, a certain relic abundance is obtained

Goal: $\Omega_X \sim 0.2$



l : very light, in complete equilibrium $\rightarrow n_l = n_l^{(e)}$

$$\frac{1}{a^3} \frac{d(n_x a^3)}{dt} = \langle \sigma v \rangle (n_x^{(0)})^2 \left[\frac{n_e n_e}{n_e^{(0)} n_x^{(0)}} - \frac{n_x n_x}{n_x^{(0)} n_x^{(0)}} \right] =$$

$$= \langle \sigma v \rangle \left[(n_x^{(0)})^2 - n_x^2 \right]$$

Recall that $T \propto a^{-1}$, so $\frac{1}{a^3} \frac{d(n_x a^3)}{dt} = \frac{1}{a^3} \frac{d}{dt} \left(\frac{n_x}{T^3} a^{3T^3} \right) \xrightarrow{\text{const}} = \frac{d}{dt} \left(\frac{n_x}{T^3} \right)$

$$= T^3 \frac{d}{dt} \left(\frac{n_x}{T^3} \right) \quad \text{Define } Y \equiv \frac{n_x}{T^3}$$

$$\Rightarrow T^3 \frac{dY}{dt} = T^3 \langle \sigma v \rangle (Y_{\text{Eq}}^2 - Y^2) \quad Y_{\text{Eq}} \equiv \frac{n_x^{(0)}}{T^3}$$

$$\Rightarrow \boxed{\frac{dY}{dt} = \langle \sigma v \rangle (Y_{\text{Eq}}^2 - Y^2)}$$

New time variable $X \equiv \frac{m}{T}$

- $x \ll 1$ high temperature, $Y \sim Y_{\text{Eq}}$
- $x \sim 1$ $T \sim m$ freeze-out
- $x \gg 1$ $T \ll m$ $Y_{\text{Eq}} \sim e^{-x}$

$$\frac{dx}{dt} = \frac{d}{dt} \left(\frac{m}{T} \right) = m \frac{d}{dt} \left(\frac{1}{T} \right) = m \frac{d}{dt} \left(\frac{a}{aT} \right) = \frac{m}{aT} \dot{a} = \frac{m}{T} \frac{\dot{a}}{a} = Hx$$

$$\frac{d}{dt} = \frac{d}{dx} \frac{dx}{dt} = Hx \frac{d}{dx}$$

$$Hx \frac{dY}{dx} = T^3 \langle \sigma v \rangle (Y_{\text{Eq}}^2 - Y^2) = \frac{m^3}{x^3} \langle \sigma v \rangle (Y_{\text{Eq}}^2 - Y^2)$$

$$\Rightarrow \frac{dY}{dx} = \frac{m^3 \langle \sigma v \rangle}{Hx^4} (Y_{\text{Eq}}^2 - Y^2) = - \frac{m^3 \langle \sigma v \rangle}{Hx^4} (Y^2 - Y_{\text{Eq}}^2) \quad \text{note } H = H(T) \text{ (not } H(x))!$$

DM production typically deep in radiation-dominated era

$$\rightarrow \rho \propto T^4, \quad H^2 \propto \rho \rightarrow H \propto \rho^{1/2} \propto T^2 \propto \frac{m^2}{x^2} \rightarrow H = \frac{H(m)}{x^2} \quad H(m) = H(x=1)$$

$$\frac{dY}{dx} = - \frac{n^3 \langle \sigma v \rangle}{H(x)^4} (Y^2 - Y_{\text{eq}}^2) = - \frac{n^3 \langle \sigma v \rangle}{H(x)^2 x^2} (Y^2 - Y_{\text{eq}}^2) = - \frac{\lambda}{x^2} (Y^2 - Y_{\text{eq}}^2)$$

or $\left[\frac{dY}{dx} = - \frac{\lambda}{x^2} (Y^2 - Y_{\text{eq}}^2) \right]$ $\lambda \equiv \frac{n^3 \langle \sigma v \rangle}{H(x)}$ Riccati equation

Assume $\lambda = \text{const}$ (not always true, but little difference)

Riccati equation has no analytical solution, let's study some limits

Final freeze-out abundance: $T \rightarrow 0$, $x \rightarrow \infty$

$$Y_{\infty} \equiv Y(x \rightarrow \infty)$$

LHS $\sim Y$ for $m \sim T$, $x \sim 1$ RHS $\sim Y^2 \lambda$

λ quite large, so unless Y very small, need RHS=0 $\rightarrow Y = Y_{\text{eq}}$

After freeze-out $x \gg 1$, $Y \gg Y_{\text{eq}}$

$$\frac{dY}{dx} \approx - \frac{\lambda Y^2}{x^2} \Rightarrow \int_{Y_f}^{Y_{\infty}} \frac{dY}{Y^2} = - \lambda \int_{x_f}^{\infty} \frac{dx}{x^2} \Rightarrow \frac{1}{Y_f} - \frac{1}{Y_{\infty}} = - \frac{\lambda}{x_f}$$

$$\Rightarrow \frac{1}{Y_{\infty}} - \frac{1}{Y_f} \approx \frac{\lambda}{x_f} \Rightarrow Y_{\infty} \approx \frac{x_f}{\lambda}$$

\uparrow
 $Y_f \gg Y_{\infty}$

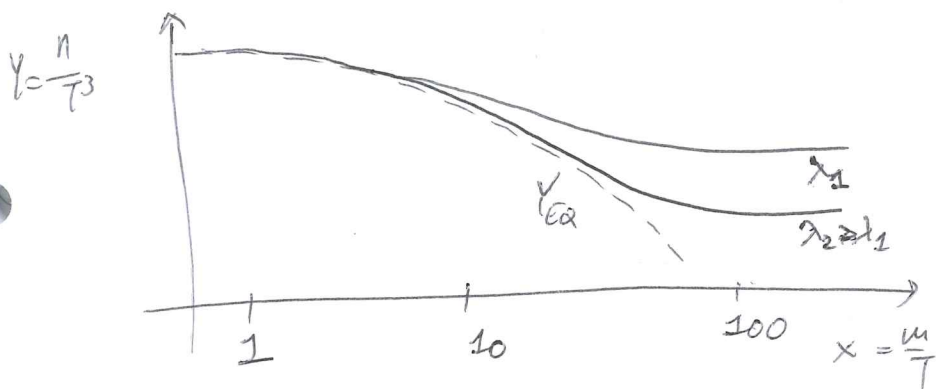
Typically $x_f \sim 20$ so freeze-out at $\frac{m}{T} \sim 20 \rightarrow T \sim \frac{m}{20}$

$$Y_{\infty} \approx \frac{x_f}{\lambda} = \frac{m}{\lambda T_f} \approx \frac{20}{\lambda}$$

makes sense!

$$= \frac{n_{x,10}}{T_0^3} \approx \frac{m}{\lambda T_f} \rightarrow n_{x,10} \approx \frac{m T_0^3}{\lambda T_f} \approx \frac{40 T_0^3}{\lambda}$$

In general need to solve for $Y(x)$ numerically



After freeze-out relic density falls as a^{-3}

late time such that $Y \sim Y_0$

$$\rho_x = m \left(\frac{a_1}{a_0} \right)^3 n_x = m Y_0 T_0^3 \left(\frac{a_1 T_1^3}{a_0 T_0^3} \right)^3 \approx \frac{m Y_0 T_0^3}{30}$$

$\uparrow = Y_0 T_1^3$

between $\sim \text{GeV}$ and $\sim \text{TeV}$
entropy injection

$$\Omega_x = \frac{\rho_x}{\rho_{cr}} \approx \frac{m Y_0 T_0^3}{30 \rho_{cr}} = \frac{m x_f T_0^3}{\lambda \cdot 30 \rho_{cr}} = \frac{H(m) x_f T_0^3}{30 m^2 \langle \sigma v \rangle \rho_{cr}}$$

$\uparrow = \frac{H(m) x_f T_0^3}{30 m^2 \langle \sigma v \rangle \rho_{cr}}$

We need $H(m) = H(x=1) - H(T \sim m)$

$$\lambda = \frac{m^3 \langle \sigma v \rangle}{H(m)}$$

Recall $H^2 = \frac{8\pi G}{3} \rho \Rightarrow H = \left(\frac{8\pi G}{3} \rho \right)^{1/2} = \left(\frac{4\pi^3 G g_*(m)}{45} m^4 \right)^{1/2}$

$\uparrow \rho = \frac{\pi^2}{30} g_*(T) T^4$

$$= \left(\frac{4\pi^3 G g_*(m)}{45} \right)^{1/2} m^2$$

$$\Rightarrow \Omega_x \approx \left(\frac{4\pi^3 G g_*(m)}{45} \right)^{1/2} \frac{x_f T_0^3}{30 \langle \sigma v \rangle \rho_{cr}}$$

Does not depend on m !
(except for g_* to be evaluated when $T \sim m$)

Want $\Omega_x \sim 0.3$ At $T \sim 200 \text{ GeV}$, $g_*(m) \sim 100$

$$\Omega_x = 0.3 h^{-2} \left(\frac{x_f}{T_0} \right) \left(\frac{g_*(m)}{100} \right)^{1/2} \frac{10^{-39} \text{ cm}^2}{\langle \sigma v \rangle}$$

weak-scale cross-section

"WIMP" miracle! Many theories give Ω_{DM} , well motivated.