

THE BOLTZMANN EQUATIONS

Recall what we saw so far: integrated version of Boltzmann equation

$$\frac{1}{a^3} \frac{d(na^3)}{dt} = \int \frac{d^3p}{(2\pi)^3 2E} \dots \delta^3(\Sigma \vec{p}) \delta(\Sigma E) |\mu|^2 * F(t)$$

$1+2 \rightarrow 3+4$ ↓ approximations

$$\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right)$$

↓ equilibrium, same equation

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}$$

$$n \sim \int d^3p f(p)$$

contains integrated information

We want a more "primitive" version to get information about f

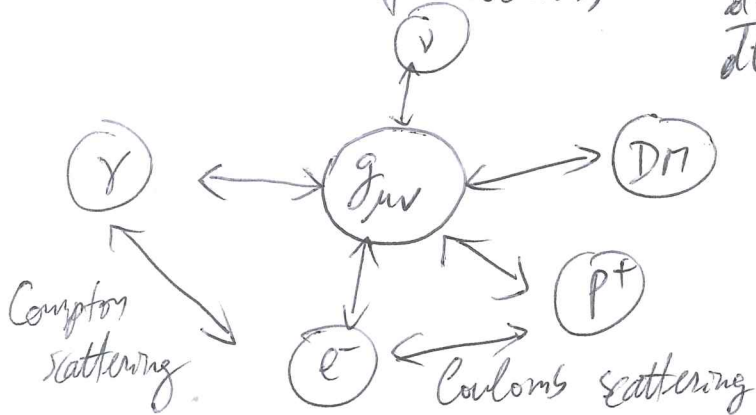
Schematically

$$\frac{df}{dt} = C[f]$$

↑ collision term

In the absence of collisions

$\frac{df}{dt} = 0 \rightarrow$ number of particles in a given phase space element is constant, but phase space element themselves move non-trivially!



Boltzmann equations for harmonic oscillator

Start from a simple example (similar to QM, simpler algebra)

$$E = \frac{p^2}{2m} + \frac{1}{2} kx^2$$

Equations of motion

$$\frac{dx}{dt} = \frac{p}{m}$$

$$\frac{dp}{dt} = -kx$$

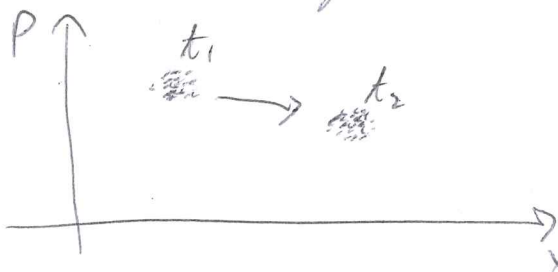
$$\frac{df}{dt} = 0$$

↳ eventually will become
Liouville equation

↑ total derivative: not so innocuous!

$$\frac{df(t,x,p)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt} = 0$$

conservation of particle number in phase space element does not
mean that f "stays still" in phase space



phase space elements move
keeping the same "shape",

number of particles unchanged

~~$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt}$$~~

$$\left(\frac{\partial f}{\partial t} + \frac{p}{m} \frac{\partial f}{\partial x} - kx \frac{\partial f}{\partial p} = 0 \right)$$

how quickly
oscillator moves
in real space

how quickly particles
lose momentum

Consider equilibrium

distribution

$$\frac{\partial f}{\partial t} = 0 \rightarrow \frac{p}{m} \frac{\partial f}{\partial x} = kx \frac{\partial f}{\partial p}$$

General solution

$$f(p,x) = f_{eq}(E)$$

f_{eq} at equilibrium only function of energy

Check: $\frac{p}{m} \frac{\partial f(E)}{\partial x} - k_x \frac{\partial f(E)}{\partial p} = \frac{p}{m} \frac{\partial f}{\partial E} \frac{\partial E}{\partial x} - k_x \frac{\partial f}{\partial E} \frac{\partial E}{\partial p} =$

$$= \frac{df}{dE} \left[\frac{p}{m} \frac{\partial E}{\partial x} - k_x \frac{\partial E}{\partial p} \right] = \frac{df}{dE} \left[\cancel{\frac{p}{m} k_x} - \cancel{k_x \frac{p}{m}} \right] = 0$$

In the presence of collisions, for there to be equilibrium these must vanish $\rightarrow f_{EQ}$ driven to equilibrium distribution, e.g.

Maxwell-Boltzmann $\propto e^{-\frac{E}{T}}$

Collisionless Boltzmann equation for photons

Let's consider $\frac{df}{dt}$ for (massless) γ . Need to specify metric and perturbations around smooth universe

Zeroth order

$$g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t)) \quad \text{no } \bar{x} \text{ dependence!}$$

$$g_{00}^{(\bar{x},t)} = -1 \quad g_{0i}(\bar{x},t) = 0 \quad g_{ij}(\bar{x},t) = \delta_{ij} a^2(t)$$

Now let's perturb it: need two more functions of space and

time $\psi(\bar{x},t)$ $\Phi(\bar{x},t)$

$$g_{00}^{(\bar{x},t)} = -1 - 2\psi(\bar{x},t)$$

$$g_{0i}(\bar{x},t) = 0$$

$$g_{ij}(\bar{x},t) = a^2 \delta_{ij} (1 + 2\Phi(\bar{x},t))$$

Very useful to know

$$(1+x)^a \approx 1 + ax$$

$$(1-x)^a \approx 1 - ax$$

$$ds^2 = -(1+2\psi)dt^2 + a^2(t) \left[(1+2\Phi) \delta_{ij} dx^i dx^j \right] = a^2(t) \left[-(1+2\psi)dt^2 + (1+2\Phi) \delta_{ij} dx^i dx^j \right]$$

ψ, Φ are small : we will drop all quadratic terms.

ψ : Newtonian potential

Φ : perturbation to spatial curvature

Sign convention:
overdense region
 $\psi < 0, \Phi > 0$

Notes (technical, can gloss over for now):

- ψ, Φ only describe scalar perturbations

Helmholtz decomposition

$g_{\mu\nu}$ 16 components $\xrightarrow[\substack{\text{symmetric} \\ g_{\mu\nu} = g_{\nu\mu}}]{}$ 10 independent components

4 scalar
2 (divergence-free) vector
(traceless, symmetric) tensor

transforms as

most general
linearized
perturbation

$4 \times 1 + 2 \times 2 + 1 \times 4 = 10$ dof

gauge invariance
remove 4
components

6 independent
components of which
2 scalars

ψ, Φ

- this is a particular choice of gauge

conformal Newtonian gauge

Alternative gauge also popular in cosmological perturbation theory

$$\left[ds^2 = a^2(\eta) [-d\eta^2 + (\delta_{ij} + h_{ij}) dx^i dx^j] \right] \rightarrow \text{synchronous gauge}$$

We will work exclusively in conformal Newtonian gauge, so we need to figure out evolution equations for ψ and Φ

Need to express $\frac{d}{dt}$

Idea: $\frac{d}{dt} \approx \sum_i \frac{\partial}{\partial \alpha_i}$

~~$f = f(\bar{x}, \bar{p}, t) = f(x^M, p^M)$~~ where $x^M = (t, \bar{x})$

note: we do NOT need to specify λ explicitly

$p^M \equiv \frac{dx^M}{d\lambda}$

In principle f defined on 8-dimensional space, but we have additional constraints

Photon is massless:

$P^2 \equiv g_{\mu\nu} p^\mu p^\nu = 0 \rightarrow p^M$ has only 3 independent components

conformal Newtonian gauge: $\text{diag}[-(1+2\psi), (1+2\psi), (1+2\psi), (1+2\psi)]$

$P^2 = 0 = -(1+2\psi)(p^0)^2 + p^2 = 0$ $p^2 \equiv g_{ij} p^i p^j$ (only spatial indices)

$\Rightarrow p^0 = \frac{p}{\sqrt{1+2\psi}} \approx p(1-\psi)$ using $(1+x)^{-\frac{1}{2}} \approx 1 - \frac{x}{2}$

~~Sign convention~~

$\hookrightarrow p^0 \approx p(1-\psi)$

~~overdense region $\psi < 0 \rightarrow 1-\psi > 1$
underdense region $\psi > 0$~~

Sign convention

overdense region $\psi < 0, \Phi > 0 \Rightarrow 1-\psi > 1 \Rightarrow$ photons whose energy moving out of potential well ($\psi < 0 \rightarrow 0$)
underdense region $\psi > 0, \Phi < 0$
 \hookrightarrow redshift

$P^0 \approx p(1-\gamma)$ basically generalizes $E=pc$ to perturbed

FRLW space-time. Whenever we meet P^0 we can replace it with p ("generalized magnitude of momentum")

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial x^i} \rightarrow \text{we won't include } P^0, \text{ just } p \text{ and its (angular) direction}$$

$$\hat{p}^i = \frac{p^i}{p} \quad \delta_{ij} \hat{p}^i \hat{p}^j = 1$$

unit vector \rightarrow

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i} \frac{d\hat{p}^i}{dt}$$

Strategy:

- Work out RHS to 1st order
- Expand LHS to 1st order after perturbing
- Equate 1st order terms

Next goal: figure out $\frac{dx^i}{dt}$, $\frac{dp}{dt}$, $\frac{d\hat{p}^i}{dt}$ to first order

Important to "see" second-order terms and set them = 0 !!!

$\frac{\partial f}{\partial \hat{p}^i} \frac{d\hat{p}^i}{dt} \rightarrow$ to 1st order

$f^{(0)}$ = Box-Einstein

$$f^{(0)} = f^{(0)}(p)$$

$$\frac{\partial f^{(0)}}{\partial \hat{p}} = 0 \quad \text{no dependence on direction!}$$

so $\frac{\partial f}{\partial \hat{p}^i}$ necessarily (at least) 1st order

$\frac{d\hat{p}^i}{dt}$ also (at least) 1st order

$\frac{\partial f}{\partial \hat{p}^i} \frac{d\hat{p}^i}{dt}$ (at least) second order!

$$\hookrightarrow \frac{df}{dt} \approx \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt}$$

let's move on to $\frac{\partial f}{\partial x^i} \frac{dx^i}{dt}$

$$\text{we } p^m = \frac{dx^m}{dt}$$

$$P^M \equiv \frac{dx^M}{d\lambda} \rightarrow P^i \equiv \frac{dx^i}{d\lambda}, P^0 \equiv \frac{dt}{d\lambda}$$

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \frac{d\lambda}{dt} = \frac{dx^i}{d\lambda} \left(\frac{dt}{d\lambda} \right)^{-1} = \frac{P^i}{P^0}$$

~~$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \frac{d\lambda}{dt} = \frac{dx^i}{d\lambda} \left(\frac{dt}{d\lambda} \right)^{-1} = \frac{P^i}{P^0}$$~~

Let's use P, \hat{P}^i

$$P^0 \approx P(1-\Psi) \quad P^i \approx C \hat{P}^i \rightarrow C = ?$$

~~$$P^2 = g_{ij} P^i P^j$$~~

$$P^2 = g_{ij} P^i P^j = g_{ij} \hat{P}^i \hat{P}^j C^2 = g_{ij} a^2 (1+2\Phi) \delta_{ij} \hat{P}^i \hat{P}^j C^2 = a^2 (1+2\Phi) C^2$$

$$P^2 = a^2 (1+2\Phi) C^2 \rightarrow P = a (1+2\Phi)^{1/2} C \rightarrow C = \frac{P}{a} (1+2\Phi)^{-1/2} \approx \frac{P(1-\Phi)}{a}$$

$$P^i = C \hat{P}^i = P \hat{P}^i \frac{1-\Phi}{a}$$

$$\frac{dx^i}{dt} = \frac{P^i}{P^0} = P \hat{P}^i \frac{1-\Phi}{a} [P(1-\Psi)]^{-1} \approx \frac{P \hat{P}^i}{P \frac{a}{a}} (1-\Phi)(1+\Psi) \approx \frac{\hat{P}^i}{a} (1+\Psi-\Phi)$$

$$(1-\Psi)^{-1} \approx (1+\Psi)$$

$$(1-\Phi)(1+\Psi) = 1+\Psi-\Phi - \Psi\Phi \approx 1+\Psi-\Phi$$

2nd order

$$\frac{dx^i}{dt} \approx \frac{\hat{P}^i}{a} (1+\Psi-\Phi)$$

Does it make sense?

Recall overdensity $\Psi < 0, \Phi > 0$

↑ "v_y" photon velocity

$1+\Psi-\Phi < 1 \rightarrow$ photons slow down in an overdensity

$$\frac{\partial}{\partial x^i} \frac{dx^i}{dt}$$

$$\frac{\partial f^{(0)}}{\partial x^i} = 0$$

= 0

so

$$\frac{\partial}{\partial x^i} \neq 0$$

only at 1st order

1st order keep only

0th order

→ neglect

potentials

$$\frac{\partial}{\partial x^i} \frac{dx^i}{dt} \approx \frac{\partial}{\partial x^i} \frac{\hat{P}^i}{a}$$

↑ 1st order

What do we have so far?

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underbrace{\frac{\partial f}{\partial x^i} \frac{\dot{x}^i}{a}}_{\approx \frac{\partial f}{\partial x^i} \frac{dx^i}{dt}} + \underbrace{\frac{\partial f}{\partial p} \frac{dp}{dt} + \frac{\partial f}{\partial \dot{p}^i} \frac{d\dot{p}^i}{dt}}_{\text{second order (at least) } \rightarrow 0}$$

We need $\frac{dp}{dt} \rightarrow$ geodesic equation!

$$\frac{dp^\mu}{d\lambda} = \frac{d^2 x^\mu}{d\lambda^2} = -\Gamma_{\alpha\beta}^{\mu} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -\Gamma_{\alpha\beta}^{\mu} p^\alpha p^\beta$$

$$\frac{dp^0}{d\lambda} = -\Gamma_{\alpha\beta}^0 p^\alpha p^\beta \quad \left\{ \begin{array}{l} \frac{dp^0}{d\lambda} = \frac{dp^0}{dt} \frac{dt}{d\lambda} = p^0 \frac{dp^0}{dt} \approx p(1-\psi) \frac{d}{dt} [p(1-\psi)] \\ p^0 = \frac{dt}{d\lambda}, p^0 \approx p(1-\psi) \end{array} \right.$$

$$p(1-\psi) \frac{d}{dt} [p(1-\psi)] = -\Gamma_{\alpha\beta}^0 p^\alpha p^\beta \Rightarrow \frac{d}{dt} [p(1-\psi)] \approx -\frac{\Gamma_{\alpha\beta}^0 p^\alpha p^\beta}{p} \quad (1+\psi)$$

$$\uparrow \frac{1}{p(1-\psi)} \approx \frac{1+\psi}{p}$$

$$\left(\frac{d}{dt} [p(1-\psi)] = (1-\psi) \frac{dp}{dt} - p \frac{d\psi}{dt} \right)$$

$$\Rightarrow (1+\psi) \times \frac{dp}{dt} (1-\psi) = \left[p \frac{d\psi}{dt} - \frac{\Gamma_{\alpha\beta}^0 p^\alpha p^\beta}{p} (1+\psi) \right] \times (1+\psi)$$

1st order \rightarrow

$$\begin{aligned} (1+\psi)(1-\psi) &\approx 1 \\ (1+\psi)^2 &\approx 1+2\psi \\ \frac{d\psi}{dt} (1+\psi) &\approx \frac{d\psi}{dt} \end{aligned}$$

$$\frac{dp}{dt} = p \frac{d\psi}{dt} (\bar{x}, t) - \frac{\Gamma_{\alpha\beta}^0 p^\alpha p^\beta}{p} (1+2\psi) =$$

$$= p \left[\frac{\partial \psi}{\partial t} + \frac{\dot{x}^i}{a} \frac{\partial \psi}{\partial x^i} \right] - \frac{\Gamma_{\alpha\beta}^0 p^\alpha p^\beta}{p} (1+2\psi)$$

$$\underbrace{\frac{dx^i}{dt} \frac{\partial \psi}{\partial x^i}}_{\text{second order}} \rightarrow$$

So we need $\Gamma_{\alpha\beta}^{\rho} = \frac{g^{\rho\sigma} \partial_{\sigma} g_{\alpha\beta}}{2}$

$$\Gamma_{\alpha\beta}^{\rho} = \frac{g^{\rho\sigma}}{2} \left[\frac{\partial g_{\sigma\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\sigma\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\sigma}} \right]$$

$$\Gamma_{\alpha\beta}^{\rho} = \frac{g^{\rho\sigma}}{2} \left[2 \frac{\partial g_{\sigma\alpha}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\sigma}} \right] \frac{p^{\alpha} p^{\beta}}{P} = \frac{g^{\rho 0}}{2} \left[2 \frac{\partial g_{0\alpha}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^0} \right] \frac{p^{\alpha} p^{\beta}}{P}$$

symmetric in α, β $v=0$

$$g^{00} = (g_{00})^{-1} = [-1+2\Phi]^{-1} \approx -1+2\Phi$$

check $(-1-2\Phi)(-1+2\Phi) = 1 - 2\Phi + 2\Phi - 4\Phi^2 \approx 1$
2nd order

$$\frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}$$

$$\Gamma_{\alpha\beta}^{\rho} = \frac{-1+2\Phi}{2} \left[2 \frac{\partial g_{0\alpha}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial t} \right] \frac{p^{\alpha} p^{\beta}}{P} = \frac{-1+2\Phi}{2} \left[-4 \frac{\partial \Phi}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial t} \right] \frac{p^{\alpha} p^{\beta}}{P}$$

$\alpha=0, \frac{\partial g_{0\alpha}}{\partial x^{\beta}} = \frac{\partial}{\partial x^{\beta}} [-1-2\Phi] \approx -2 \frac{\partial \Phi}{\partial x^{\beta}}$

$$-\frac{\partial g_{\alpha\beta}}{\partial t} \frac{p^{\alpha} p^{\beta}}{P} = -\frac{\partial g_{00}}{\partial t} \frac{p^0 p^0}{P} - \frac{\partial g_{ij}}{\partial t} \frac{p^i p^j}{P} = \frac{2\partial\Phi}{\partial t} p - a^2 \delta_{ij} \left[2 \frac{\partial \Phi}{\partial t} + 2H(1+2\Phi) \right] \frac{p^i p^j}{P}$$

$$\frac{\partial \delta_{ij}}{\partial t} = \frac{\partial}{\partial t} [a^2 \delta_{ij} (1+2\Phi)] =$$

$$= a^2 \delta_{ij} \frac{\partial}{\partial t} [1+2\Phi] + (1+2\Phi) \delta_{ij} \frac{\partial a^2}{\partial t} = a^2 \delta_{ij} \frac{2\partial\Phi}{\partial t} + (1+2\Phi) \delta_{ij} \dot{a} 2a =$$

$$= a^2 \delta_{ij} \frac{2\partial\Phi}{\partial t} + a^2 \delta_{ij} (1+2\Phi) 2 \frac{\dot{a}}{a} = a^2 \delta_{ij} \left[\frac{2\partial\Phi}{\partial t} + 2H(1+2\Phi) \right]$$

So

$$-\frac{\partial g_{\alpha\beta}}{\partial t} \frac{p^{\alpha} p^{\beta}}{P} \approx \frac{2\partial\Phi}{\partial t} p - a^2 \delta_{ij} \left[\frac{2\partial\Phi}{\partial t} + 2H(1+2\Phi) \right] \frac{p^i p^j}{P} \approx \frac{p^i p^j}{P} \delta_{ij} \approx \frac{p^2}{a^2} (1-2\Phi)$$