$$e^- + p \longleftrightarrow H + \gamma$$
$$\;1 \quad\; 2 \qquad\quad 3 \quad 4$$

While in equilibrium, Saha equation is valid

$$\frac{n_e n_p}{n_H \cancel{n_\gamma}} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)} \cancel{n_\gamma^{(0)}}} \xRightarrow[n_\gamma = n_\gamma^{(0)}]{} \frac{n_e n_p}{n_H} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

Neutrality of the Universe $\longrightarrow$ $n_e = n_p$

Free electron fraction $\quad X_e \equiv \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H} = \frac{n_e}{n_b}$

$$1 - X_e = \frac{n_H}{n_e + n_H} = \frac{n_H}{n_p + n_H} = \frac{n_H}{n_b}$$

$$\frac{n_e n_p}{n_H} = \frac{X_e (n_e + n_H)\, X_e (n_e + n_H)}{(1 - X_e)(n_e + n_H)} = \frac{X_e^2}{1 - X_e} (n_e + n_H)$$

B : hydrogen binding energy (ε)

$$\frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{(m_e + m_p - m_H)}{T}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\varepsilon_0}{T}}$$

$$n_i^{(0)} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}$$, $m_p \sim m_H$ in pre-factor

$$\rightarrow \frac{X_e^2}{1 - X_e} = \underbrace{\frac{1}{n_e + n_H} \left[\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\varepsilon_0}{T}}\right]}$$

Denominator $\quad n_e + n_H \sim n_b = \eta_b n_\gamma \sim 10^{-9} T^3$

$T \sim \varepsilon_0 \longrightarrow RHS \sim 10^9 \left(\frac{m_e}{T}\right)^{3/2} \simeq 10^{15}$!!!

LHS = RHS only if $X_e \simeq 1$ $\Rightarrow$ no ~~H~~ hydrogen!

High η prevents H formation even when initially energetically favorable $T \sim \varepsilon_0$. Need $T \ll \varepsilon_0$ before recombination can proceed

X_e drops $\longrightarrow$ process out of equilibrium, cannot use Saha equation, need Boltzmann equation

$$\frac{1}{a^3}\frac{d(n_e a^3)}{dt} = n_e^{(0)} n_p^{(0)} \langle \sigma v \rangle \left\{ \frac{n_H \cancel{n_\gamma}}{n_H^{(0)} \cancel{n_\gamma^{(0)}}} - \frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} \right\} =$$

$$= n_e^{(0)} n_p^{(0)} \langle \sigma v \rangle \left[\frac{n_H}{n_H^{(0)}} - \frac{n_e^2}{n_e^{(0)} n_p^{(0)}} \right] =$$

$$= n_b \langle \sigma v \rangle \left[\frac{n_H}{n_b} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} - \frac{n_e^2}{n_b} \right] =$$

$n_e = X_e n_b$

$n_H = (1-X_e) n_b$

$$\frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} \approx \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\varepsilon_0}{T}}$$

$$= n_b \langle \sigma v \rangle \left[(1-X_e)\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\varepsilon_0}{T}} - X_e^2 n_b^{\cancel{2}} \right]$$

$$\frac{1}{a^3}\frac{d(n_e a^3)}{dt} = \frac{1}{a^3}\frac{d(n_b X_e a^3)}{dt} = \frac{1}{a^3} n_b a^3 \frac{dX_e}{dt} = n_b \frac{dX_e}{dt}$$

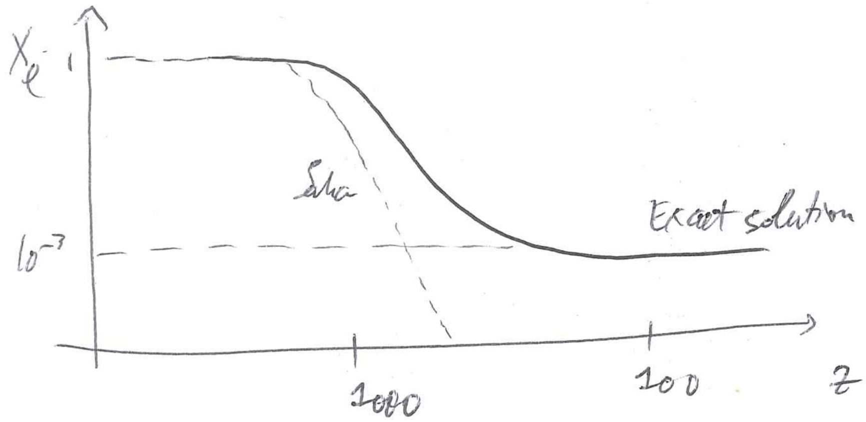
$n_b a^3 = \text{const}$

$$\Longrightarrow \boxed{\frac{dX_e}{dt} = \langle \sigma v \rangle \left[(1-X_e)\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\varepsilon_0}{T}} - X_e^2 n_b \right] = \left[(1-X_e)\beta - X_e^2 n_b \alpha^{(2)} \right]}$$

$$\beta \equiv \langle \sigma v \rangle \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\varepsilon_0}{T}}$$ Ionization rate

$$\alpha^{(2)} \equiv \langle \sigma v \rangle \simeq 9.78 \frac{\alpha^2}{m_e^2}\left(\frac{\varepsilon_0}{T}\right)^{1/2} \ln\left(\frac{\varepsilon_0}{T}\right)$$ recombination rate

Need to solve numerically for $X_e(t)$



Saha solution falls out of equilibrium, but to good approximation recombination occurs when $X_e^{(0)} \sim 10^{-2}$!

from Saha!

$z_* \sim 1000$ redshift of recombination

When $X_e \sim 10^{-2}$ photons decouple because Compton scattering becomes inefficient $(\Gamma \sim X_e n_b \sigma_T)$. So (γ) decoupling happens roughly during recombination, in fact the two terms are often used interchangeably.

<u>Dark matter freeze-out</u> (hypothetical!!!)

Weakly interacting massive particle (WIMP, $\sigma \sim \sigma_{ew} \sim 10^{-39}\,\mathrm{cm}^2$, $m \sim \mathrm{GeV}$) in thermal equilibrium at high temperatures $(XX \to \ell\ell)$

light/massless

Freeze-out: inability of annihilations to maintain equilibrium

If this particle were kept in equilibrium indefinitely, $n \sim e^{-\frac{m}{T}} \to 0$!

Because of freeze-out, a certain <u>relic abundance</u> is obtained

Goal: $\Omega_X \approx 0.2$

$$\overset{1}{X}\,\overset{2}{X} \longrightarrow \overset{3}{\ell}\,\overset{4}{\ell}$$

ℓ: very light, in complete equilibrium $\longrightarrow n_\ell = n_\ell^{(0)}$

$$\frac{1}{a^3}\frac{d(n_\chi a^3)}{dt} = \langle\sigma v\rangle (n_\chi^{(0)})^2\left[\cancel{\frac{n_\ell n_\ell}{n_\ell^{(0)} n_\ell^{(0)}}} - \frac{n_\chi n_\chi}{n_\chi^{(0)} n_\chi^{(0)}}\right] =$$

$$= \langle\sigma v\rangle\left[(n_\chi^{(0)})^2 - n_\chi^2\right]$$

Recall that $T \propto a^{-1}$, so $\frac{1}{a^3}\frac{d(n_\chi a^3)}{dt} = \frac{1}{a^3}\frac{d}{dt}\left(\frac{n_\chi}{T^3}\overbrace{a^3T^3}^{\text{const}}\right) = \frac{a^3T^3}{a^3}\frac{d}{dt}\left(\frac{n_\chi}{T^3}\right)$

$= T^3\frac{d}{dt}\left(\frac{n_\chi}{T^3}\right)$ Define $Y \equiv \frac{n_\chi}{T^3}$

$$\Longrightarrow \cancel{T^3}\frac{dY}{dt} = T^{\cancel{6}\,3}\langle\sigma v\rangle\left(Y_{EQ}^2 - Y^2\right) \qquad Y_{EQ} \equiv \frac{n_\chi^{(0)}}{T^3}$$

$$\Longrightarrow \boxed{\frac{dY}{dt} = T^3\langle\sigma v\rangle\left(Y_{EQ}^2 - Y^2\right)}$$

New time variable $x \equiv \frac{m}{T}$

- $x \ll 1$: high temperatures $Y \sim Y_{EQ}$
- $x \sim 1$: $T \sim m$ freeze-out
- $x \gg 1$: $T \ll m$ $Y_{EQ} \sim e^{-x}$

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{m}{T}\right) = m\frac{d}{dt}\left(\frac{1}{T}\right) = m\frac{d}{dt}\left(\frac{a}{\underset{\text{const}}{aT}}\right) = \frac{m}{aT}\dot{a} = \underbrace{\frac{m}{T}}_{x}\underbrace{\frac{\dot{a}}{a}}_{H} = Hx$$

$$\frac{d}{dt} = \frac{d}{dx}\frac{dx}{dt} = Hx\frac{d}{dx}$$

$$Hx\frac{dY}{dx} = T^3\langle\sigma v\rangle\left(Y_{EQ}^2 - Y^2\right) = \frac{m^3}{x^3}\langle\sigma v\rangle\left(Y_{EQ}^2 - Y^2\right)$$

$$\Longrightarrow \frac{dY}{dx} = \frac{m^3\langle\sigma v\rangle}{Hx^4}\left(Y_{EQ}^2 - Y^2\right) = -\frac{m^3\langle\sigma v\rangle}{Hx^4}\left(Y^2 - Y_{EQ}^2\right) \qquad \text{note } H = H(T)!$$

DM production typically deep in radiation-dominated era

$\longrightarrow \rho \propto T^4$, $H^2 \propto \rho \rightarrow H \propto \sqrt{\rho} \propto T^2 \propto \frac{m^2}{x^2} \longrightarrow H = \frac{H(m)}{x^2}$ $H(m) = H(x=1)$

$$\frac{dY}{dx} = -\frac{m^3\langle\sigma v\rangle}{Hx^4}\left(Y^2 - Y_{EQ}^2\right) = -\frac{m^3\langle\sigma v\rangle}{H(m)\,x^2}\left(Y^2 - Y_{EQ}^2\right) = -\frac{\lambda}{x^2}\left(Y^2 - Y_{EQ}^2\right)$$

so $$\boxed{\frac{dY}{dx} = -\frac{\lambda}{x^2}\left(Y^2 - Y_{EQ}^2\right)}$$ $$\lambda \equiv \frac{m^3\langle\sigma v\rangle}{H(m)}$$ Riccati equation

Assume $\lambda = \text{const}$ (not always true, but little difference)

Riccati equation has no analytical solution, let's study some limits

Final freeze-out abundance : $T \to 0$, $x \to \infty$

$Y_\infty \equiv Y(x \to \infty)$

LHS $\sim Y$ for $m \sim T$, $x \sim 1$ RHS $\sim Y^2\lambda$

λ quite large, so unless Y very small, need RHS $= 0$ $\to Y = Y_{EQ}$

After freeze-out $x \gg 1$, $Y \gg Y_{EQ}$

$$\frac{dY}{dx} \simeq -\frac{\lambda Y^2}{x^2} \Longrightarrow \int_{Y_f}^{Y_\infty}\frac{dY}{Y^2} = -\lambda\int_{x_f}^{\infty}\frac{dx}{x^2} \Longrightarrow \frac{1}{Y_f} - \frac{1}{Y_\infty} = -\frac{\lambda}{x_f}$$

$$\Longrightarrow \frac{1}{Y_\infty} - \frac{1}{Y_f} \simeq \frac{\lambda}{x_f} \Longrightarrow Y_\infty \simeq \frac{x_f}{\lambda}$$

$Y_f \gg Y_\infty$

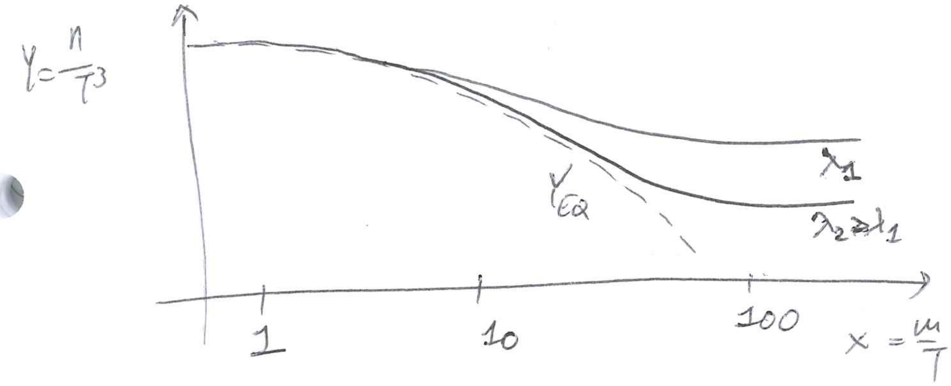
Typically $x_f \sim 10$ so freeze-out at $\frac{m}{T} \sim 10 \to T \sim \frac{m}{10}$

makes sense!

$$Y_\infty \simeq \frac{x_f}{\lambda} = \cancel{\frac{m}{\lambda T_f}} \simeq \frac{10}{\lambda}$$

$$= \frac{n_{x,0}}{T_0^3} \simeq \frac{m}{\lambda T_f} \to n_{x,0} \simeq \frac{m T_0^3}{\lambda T_f} \simeq \frac{10 T_0^3}{\lambda}$$

In general need to solve for $Y(x)$ numerically



After freeze-out relic density falls a, a^{-3}

late time such that $Y \sim Y_\infty$

$$\rho_x = m\left(\frac{a_1}{a_0}\right)^3 n_x = m Y_\infty T_0^3 \left(\frac{a_1 T_1}{a_0 T_0}\right)^3 \simeq \frac{m Y_\infty T_0^3}{30}$$

$= Y_\infty T_1^3$

between ~keV and ~MeV entropy injection

$$\Omega_x = \frac{\rho_x}{\rho_{cr}} \simeq \frac{m Y_\infty T_0^3}{30 \rho_{cr}} = \frac{m x_f}{\lambda} \frac{T_0^3}{30\rho_{cr}}$$

$$= \frac{H(m) x_f T_0^3}{30 m^2 \langle \sigma v \rangle \rho_{cr}}$$

$\lambda \equiv \frac{m^3 \langle \sigma v\rangle}{H(m)}$

We need $H(m) = H(x=1) = H(T \sim m)$

Recall $H^2 = \frac{8\pi G}{3}\rho \Rightarrow H = \left(\frac{8\pi G}{3}\rho\right)^{1/2} = \left(\frac{4\pi^3 G g_*(m)}{45} m^4\right)^{1/2} =$

$\rho = \frac{\pi^2}{30} g_*(T) T^4$

$$= \left(\frac{4\pi^3 G g_*(m)}{45}\right)^{1/2} m^2$$

$$\Rightarrow \Omega_x \simeq \left(\frac{4\pi^3 G g_*(m)}{45}\right)^{1/2} \frac{x_f T_0^3}{30 \langle \sigma v\rangle \rho_{cr}}$$

does not depend on m! (except for x_f, g_* to be evaluated when $T \sim m$)

Want $\Omega_x \sim 0.3$ At $T \sim 100$ GeV, $g_*(m) \sim 100$

$$\Omega_x = 0.3 h^{-2} \left(\frac{x_f}{10}\right)\left(\frac{g_*(m)}{100}\right)^{1/2} \frac{10^{-37}\,\text{cm}^2}{\langle \sigma v\rangle}$$

weak-scale cross-section

"WIMP" miracle! Many theories give $\sigma \sim \sigma_W$, well motivated.