

$$\approx 2 \frac{\partial \psi}{\partial t} p - 2 a^2 \delta_{ij} \frac{\partial \Phi}{\partial t}$$

$$(1+2\psi)(1-2\Phi) \approx 1$$

$$(1-2\Phi) \frac{\partial \Phi}{\partial t} \approx \frac{\partial \Phi}{\partial t}$$

$$\bullet \approx 2 \frac{\partial \psi}{\partial t} p - \frac{a^2}{a^2} (1-2\Phi) \left[2 \frac{\partial \Phi}{\partial t} + 2H(1+2\Phi) \right] \approx 2 \frac{\partial \psi}{\partial t} p - 2p \frac{\partial \Phi}{\partial t} - 2Hp$$

Putting everything together

$$p^\alpha_{\alpha\beta} \frac{p^\alpha p^\beta}{p} = \frac{-1+2\psi}{2} \left[2 \frac{\partial g_{\alpha\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial t} \right] \frac{p^\alpha p^\beta}{p} \approx \frac{-1+2\psi}{2} \left[-4 \frac{\partial \psi}{\partial x^\beta} p^\beta + 2p \frac{\partial \psi}{\partial t} - 2p \frac{\partial \Phi}{\partial t} - 2Hp \right] \approx$$

$$\frac{2 \partial g_{\alpha\alpha}}{\partial x^\beta} \frac{p^\alpha p^\beta}{p} = -4 \frac{\partial \psi}{\partial x^\beta} \frac{p^\alpha p^\beta}{p} \approx -4 \frac{\partial \psi}{\partial x^\beta} \frac{p(1-\psi) p^\beta}{p} \approx -4 \frac{\partial \psi}{\partial x^\beta} p^\beta \quad \frac{\partial \psi}{\partial x^\beta} (1-\psi) \approx \frac{\partial \psi}{\partial x^\beta}$$

$$\approx \frac{-1+2\psi}{2} \left[-4 \left(\frac{\partial \psi}{\partial t} p + \frac{\partial \psi}{\partial x^i} \frac{p \hat{p}^i}{a} \right) + 2p \frac{\partial \psi}{\partial t} - p \left(2 \frac{\partial \Phi}{\partial t} + H \right) \right] \approx$$

$$\frac{\partial \psi}{\partial x^\alpha} p^\alpha \equiv \sum_p \frac{\partial \psi}{\partial x^\alpha} p^\alpha = \frac{\partial \psi}{\partial t} p^0 + \frac{\partial \psi}{\partial x^i} p^i \approx \frac{\partial \psi}{\partial t} p(1-\psi) + \frac{\partial \psi}{\partial x^i} \frac{p \hat{p}^i}{a} (1-\psi) \approx \frac{\partial \psi}{\partial t} p + \frac{\partial \psi}{\partial x^i} \frac{p \hat{p}^i}{a}$$

$$\approx \frac{-1+2\psi}{2} \left[-4p \frac{\partial \psi}{\partial t} - 4 \frac{\partial \psi}{\partial x^i} \frac{p \hat{p}^i}{a} + 2p \frac{\partial \psi}{\partial t} - 2p \frac{\partial \Phi}{\partial t} - 2Hp \right] =$$

$$= \frac{-1+2\psi}{2} \left[-p \frac{\partial \psi}{\partial t} - 2 \frac{\partial \psi}{\partial x^i} \frac{p \hat{p}^i}{a} - p \left(\frac{\partial \Phi}{\partial t} + H \right) \right]$$

Recall where we started

$$\frac{dp}{dt} = p \left(\frac{\partial \psi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \psi}{\partial x^i} \right) - \frac{p^\alpha p^\beta}{p} (1+2\psi) = p \left(\frac{\partial \psi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \psi}{\partial x^i} \right) - p \frac{\partial \psi}{\partial t} - 2 \frac{\partial \psi}{\partial x^i} \frac{p \hat{p}^i}{a} - p \left(\frac{\partial \Phi}{\partial t} + H \right)$$

$$(-1+2\psi)(1+2\psi) \approx -1$$

$$\Rightarrow \boxed{\frac{1}{p} \frac{dp}{dt} = -H - \frac{\partial \Phi}{\partial t} - \frac{\hat{p}^i}{a} \frac{\partial \psi}{\partial x^i}}$$

change in momentum of photon moving through perturbed FLRW universe

Physical interpretation:

• overdense region $\psi < 0, \Phi > 0$

-H: loss of momentum due to Hubble expansion (redshift)

- $\frac{\partial \Phi}{\partial t} > 0$: δ in deepening gravitational well loses energy (redshift)

- $\frac{\partial \psi}{\partial x^i} \neq 0$: δ travelling into a well gains energy, redshifted when leaving well

Big picture: return to full Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p} \frac{dp}{dt}$$

$$\left[\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial f}{\partial x^i} - p \frac{\partial f}{\partial p} \left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right] \right] = 0$$

(Collisionless)
Boltzmann equations for photons

need to perturb and expand!

(integrated) continuity equation

(integrated) Euler equation

how photons lose energy in ~~their~~ expanding universe and in the presence of under/overdensities

To make progress, expand $f(\bar{x}, \hat{p}, t) = f(\bar{x}, p, \hat{p}, t)$ around $f^{(0)}$ [LHS]

$$f^{(0)} = \frac{1}{\exp\left[\frac{p}{T(t)}\right] - 1}$$

~~$f(\bar{x}, \hat{p}, t) = \frac{1}{\exp\left[\frac{p}{T(t)[1+\Theta(\bar{x}, \hat{p}, t)]}\right] - 1}$~~

$$f = \left[\exp\left\{ \frac{p}{T(t)[1+\Theta(\bar{x}, \hat{p}, t)]} \right\} - 1 \right]^{-1}$$

zero-order temperature $T(t)$ function of time only $T \propto a^{-1}$

$\Theta = \frac{\delta T}{T}$ perturbation to the distribution function

$\Theta(\bar{x}, \hat{p}, t)$
 → anisotropies
 → like zero-order temperature
 → inhomogeneities

no p dependence: see later!

~~Compton scattering~~ ~~leaves~~ ~~distribution~~ ~~unchanged~~

$\Theta \ll 1$, so expand in small Θ

useful property of Bose-Einstein distribution $T \frac{df^{(0)}}{dT} = -p \frac{df^{(0)}}{dp}$ easy to check!

Check: $f^{(0)} = \left[e^{\frac{p}{T}} - 1 \right]^{-1}$

$$\frac{\partial f^{(0)}}{\partial T} = -\frac{1}{(e^{\frac{p}{T}} - 1)^2} \times -\frac{p}{T^2} e^{\frac{p}{T}} =$$

$$= \frac{e^{\frac{p}{T}} p}{T^2 (e^{\frac{p}{T}} - 1)^2}$$

$$\frac{\partial f^{(0)}}{\partial p} = -\frac{1}{(e^{\frac{p}{T}} - 1)^2} \frac{1}{T} e^{\frac{p}{T}} =$$

$$= -\frac{e^{\frac{p}{T}}}{T (e^{\frac{p}{T}} - 1)^2}$$

$$T \frac{\partial f^{(0)}}{\partial T} = -\frac{p \partial f^{(0)}}{\partial p}$$

Expand $f \approx f^{(0)} + \delta f$ up to first order

$$f \approx \left[e^{\frac{p}{T}} - 1 \right]^{-1} + \frac{\partial f^{(0)}}{\partial T} T \Theta = f^{(0)} + \frac{\partial f^{(0)}}{\partial T} T \Theta = f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta$$

$$f \approx f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta$$

↑ perturbed distribution function

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \hat{p}^i \frac{\partial f}{\partial x^i} - p \frac{\partial f}{\partial p} \left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right]$$

Look at 0th and 1st order terms

0th order

~~$$\frac{\partial f^{(0)}}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial f^{(0)}}{\partial x^i} - p \frac{\partial f^{(0)}}{\partial p} \left[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right] = 0$$~~

$f^{(0)}$ doesn't depend on \bar{x}

1st order

1st order

$$\left[\frac{\partial f^{(0)}}{\partial t} - H p \frac{\partial f^{(0)}}{\partial p} \right] \hat{=} 0 \equiv \text{zeroth order (at zeroth order collisions vanish)}$$

$$\frac{\partial f^{(0)}}{\partial t} - H_p \frac{\partial f^{(0)}}{\partial p} = 0$$

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial T} \frac{dT}{dt} = -\frac{dT}{dt} \frac{p}{T} \frac{\partial f^{(0)}}{\partial p} = \left(-\frac{dT/dt}{T}\right) p \frac{\partial f^{(0)}}{\partial p}$$

$$\Rightarrow \left(-\frac{dT/dt}{T} - \frac{da/dt}{a}\right) p \frac{\partial f^{(0)}}{\partial p} \Rightarrow \frac{1}{T} \frac{dT}{dt} = -\frac{1}{a} \frac{da}{dt} \Rightarrow \int \frac{dT}{T} = -\int \frac{da}{a}$$

$$\Rightarrow \ln T = \ln(a^{-1}) + C \Rightarrow \boxed{T \propto \frac{1}{a}}$$

Boltzmann equations at zeroth order confirm photon redshift !!!

1st order

$$\frac{df}{dt} \Big|_{1st\ order} = \frac{\partial}{\partial t} \left[f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta \right] + \frac{\hat{p}^i}{a} \frac{\partial}{\partial x^i} \left[f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta \right] - H_p \left[p \frac{\partial f^{(0)}}{\partial p} - p \frac{\partial f^{(0)}}{\partial p} \Theta \right]$$

$$- p \frac{\partial \Phi}{\partial p} \frac{\partial}{\partial p} \left[f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta \right] - \frac{p \hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \frac{\partial}{\partial p} \left[f^{(0)} - p \frac{\partial f^{(0)}}{\partial p} \Theta \right]$$

1st order pieces

$$\frac{df}{dt} \Big|_{1st\ order} = -p \frac{\partial}{\partial t} \left[\frac{\partial f^{(0)}}{\partial p} \Theta \right] - \frac{p \hat{p}^i}{a} \frac{\partial \Theta}{\partial x^i} \frac{\partial f^{(0)}}{\partial p} + H_p \Theta \frac{\partial}{\partial p} \left[p \frac{\partial f^{(0)}}{\partial p} \right]$$

$$-p \frac{\partial f^{(0)}}{\partial p} \left[\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right]$$

$(\Phi, \Psi) \times \Theta$ 2nd order

$$\Rightarrow -p \frac{\partial}{\partial t} \left[\frac{\partial f^{(0)}}{\partial p} \Theta \right] = -p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \Theta}{\partial t} - p \Theta \frac{\partial^2 f^{(0)}}{\partial t \partial p} = -p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \Theta}{\partial t} - p \Theta \frac{\partial}{\partial t} \frac{\partial f^{(0)}}{\partial p}$$

$$= -p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \Theta}{\partial t} + p \Theta \frac{dT/dt}{T} \frac{\partial}{\partial p} \left[p \frac{\partial f^{(0)}}{\partial p} \right]$$

$$\frac{\partial f^{(0)}}{\partial T} = -\frac{p}{T} \frac{\partial f^{(0)}}{\partial p}$$

$$\frac{1}{T} \frac{dT}{dt} = -\frac{1}{a} \frac{da}{dt} = -H$$

$$\left. \frac{df}{dt} \right|_{1st\ order} = -p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \Theta}{\partial t} + p \Theta \frac{dT/dt}{T} \frac{\partial}{\partial p} \left[p \frac{\partial f^{(0)}}{\partial p} \right] - p \hat{p}^i \frac{\partial \Theta}{\partial x^i} \frac{\partial f^{(0)}}{\partial p}$$

$$+ H p \Theta \frac{\partial}{\partial p} \left[p \frac{\partial f^{(0)}}{\partial p} \right] - p \frac{\partial f^{(0)}}{\partial p} \left[\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right]$$

$$\Rightarrow \left. \frac{df}{dt} \right|_{first\ order} = -p \frac{\partial f^{(0)}}{\partial p} \left[\underbrace{\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Theta}{\partial x^i}}_{\text{free-streaming}} + \underbrace{\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i}}_{\text{gravity}} \right]$$

Note: "x" always appears in the combination "ax" physical distance

first order per Boltzmann equation for photons (still collisionless)

free-streaming

gravity

Collision terms: Compton scattering (MESSY!)



Schematically $C[f(\bar{p})] = \sum_{\bar{q}, \bar{q}', \bar{p}'} |M|^2 \{ f_e(\bar{q}') f(\bar{p}') - f_e(\bar{q}) f(\bar{p}) \}$

Neglected Bose enhancement and Pauli blocking (fine at 1st order)

+ conservation of momentum: $\bar{p} + \bar{q} = \bar{p}' + \bar{q}'$

+ conservation of energy: $E(p) + E_e(q) = E(p') + E_e(q')$

Collision term

$$C[f(\vec{p})] = \frac{1}{P} \int \frac{d^3\vec{q}}{(2\pi)^3 2E_q} \int \frac{d^3\vec{q}'}{(2\pi)^3 2E_{q'}} \int \frac{d^3\vec{p}'}{(2\pi)^3 2E_{p'}} |M|^2 (2\pi)^4$$

$$\times \delta^3(\vec{p} + \vec{q} - \vec{p}' - \vec{q}') \delta[E(p) + E_q - E(p') - E_{q'}] \{f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\}$$

$$E(p) = p \quad E(p') = p' \quad E_q \approx m_e + \frac{q^2}{2m_e} \quad E_{q'} \approx m_e + \frac{q'^2}{2m_e}$$

Similar to collision term seen earlier, but not integrated over \vec{p}

since we are interested in following $f(\vec{p})$

Use δ^3 to do \vec{q}' integral $\rightarrow \vec{q}' = \vec{p} + \vec{q} - \vec{p}'$

Approximate $E_q \approx m_e$ $E_{q'} \approx m_e$

$$C[f(\vec{p})] \approx \frac{1}{P} \int \frac{d^3\vec{q}}{(2\pi)^3 2m_e} \int \frac{d^3\vec{p}'}{(2\pi)^3 2p'} \int \frac{d^3\vec{q}'}{(2\pi)^3 2m_e} (2\pi)^4 \delta^3(\vec{p} + \vec{q} - \vec{p}' - \vec{q}') \delta(E(p) + E_q - E(p') - E_{q'})$$

$$\times \delta\left[p + m_e + \frac{q^2}{2m_e} - p' - m_e - \frac{q'^2}{2m_e}\right] |M|^2 \{f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\}$$

$$= \frac{1}{4m_e^2 p} \int \frac{d^3\vec{q}}{(2\pi)^3} \int \frac{d^3\vec{p}'}{(2\pi)^3 p'} \int \frac{d^3\vec{q}'}{(2\pi)^3} (2\pi)^4 \delta^3(\dots) \delta(\dots) |M|^2 \{f_e(\vec{q}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\}$$

$\vec{q}' \Rightarrow \vec{p} + \vec{q} - \vec{p}'$

$$\approx \frac{\pi}{4m_e^2 p} \int \frac{d^3\vec{q}}{(2\pi)^3} \int \frac{d^3\vec{p}'}{(2\pi)^3 p'} \delta\left[p + \frac{q^2}{2m_e} - p' - \frac{(\vec{q} + \vec{p} - \vec{p}')^2}{2m_e}\right] |M|^2 \{f_e(\vec{q} + \vec{p} - \vec{p}')f(\vec{p}') - f_e(\vec{q})f(\vec{p})\}$$

Kinematics of non-relativistic Compton scattering: very little energy is transferred!

$$E_q - E_q(\vec{q} + \vec{p} - \vec{p}') = \frac{q^2}{2m_e} - \frac{(\vec{q} + \vec{p} - \vec{p}')^2}{2m_e} \approx \frac{q^2}{2m_e} - \frac{q^2}{2m_e} - \frac{p^2}{2m_e} - \frac{p'^2}{2m_e} + \frac{(\vec{p}' - \vec{p}) \cdot \vec{q}}{m_e}$$

$$\approx \frac{(\vec{p}' - \vec{p}) \cdot \vec{q}}{m_e} \quad \text{as } q \gg p, p'$$

Nearly elastic scattering $p' \approx p$

so $\vec{p}' - \vec{p} \sim O(p) \sim O(T)$

$E_e(q) - E_e(\vec{q} + \vec{p} - \vec{p}') \approx \frac{(\vec{p}' - \vec{p}) \cdot \vec{q}}{m_e} \sim T v_b \left(T \frac{q}{m_e} \right)$

So change in electron energy due to Compton scattering $\sim T v_b$

* Kinetic energy of electron $\sim T \Rightarrow$ Fractional energy change in a single Compton collision small, $O(v_b)$

So expand final electron kinetic energy around zeroth order value

$\frac{(\vec{q} + \vec{p} - \vec{p}')^2}{2m_e} \approx \frac{q^2}{2m_e}$

Expand δ (tricky?)

$\delta \left[p + \frac{q^2}{2m_e} - p' - \frac{(\vec{q} + \vec{p} - \vec{p}')^2}{2m_e} \right] \approx \delta(p - p') + (E_e(q') - E_e(q)) \frac{\partial \delta(p + E_e(q) - p' - E_e(q'))}{\partial E_e(q')}$

$\approx \delta(p - p') + \frac{(\vec{p} - \vec{p}') \cdot \vec{q}}{m_e} \frac{\partial \delta(p - p')}{\partial p'}$

Used $\frac{\partial f(x-y)}{\partial x} = - \frac{\partial f(x-y)}{\partial y}$

$\rightarrow f_e(\vec{q} + \vec{p} - \vec{p}') f(\vec{p}') - f_e(\vec{q}) f(\vec{p}) \approx f_e(\vec{q}) f(\vec{p}') - f_e(\vec{q}) f(\vec{p}) = f_e(\vec{q}) [f(\vec{p}') - f(\vec{p})]$

$f_e(\vec{q} + \vec{p} - \vec{p}') \approx f_e(\vec{q})$

Putting everything together...

$C[\rho(\vec{p})] = \frac{\pi}{4m_e^2 p} \int d^3q \frac{f_e(\vec{q})}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3 p'} |M|^2 \left\{ \delta(p - p') + \frac{(\vec{p} - \vec{p}') \cdot \vec{q}}{m_e} \frac{\partial \delta(p - p')}{\partial p'} \right\} \{ f(\vec{p}') - f(\vec{p}) \}$

Later integrate out $\frac{\partial \delta(p - p')}{\partial p'}$ by parts...

$\int A \frac{\partial \delta}{\partial p'} = A \delta - \int \frac{\partial A}{\partial p'} \delta$

$|M|^2 \approx 8\pi\sigma_T^2 m_e^2$ σ_T : Thomson cross-section

↳ Ignores angular dependence (~~important for polarization~~)

$\lesssim 1\%$ effect in collision term $\propto (1 + \cos^2[\hat{p} \cdot \hat{p}'])$

↳ Ignores polarization $\propto |\hat{\epsilon} \cdot \hat{\epsilon}'|^2$

↳ CMB photons are polarized at a very small level

Important, but we ignore it here

So to recap

$$C[f(\vec{p})] \approx \frac{\pi}{4m_e^2 p} \int d^3q \frac{f_e(\vec{q})}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3 p'} 8\pi\sigma_T m_e^2 \left\{ \delta(\vec{p}-\vec{p}') + \frac{(\vec{p}-\vec{p}') \cdot \vec{q}}{m_e} \frac{\partial \delta(\vec{p}-\vec{p}')}{\partial p'} \right\} [f(\vec{p}') - f(\vec{p})]$$

Keep only terms first order in energy transfer ($\sim v_b$) note: $\frac{q}{m_e} \sim v_b$

Expand this mess...

$$C[f(\vec{p})] = \frac{2\pi^2 \sigma_T n_e}{p} \int \frac{d^3p'}{(2\pi)^3 p'} \left[\delta(\vec{p}-\vec{p}') + (\vec{p}-\vec{p}') \cdot \vec{v}_b \frac{\partial \delta(\vec{p}-\vec{p}')}{\partial p'} \right] \times \left\{ f(\vec{p}')^{(0)} - f(\vec{p})^{(0)} - p' \frac{\partial f^{(0)}}{\partial p'} \Theta(\hat{p}') + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}) \right\} \approx$$

$\int \delta(\vec{p}-\vec{p}') [f(\vec{p}') - f(\vec{p})] = 0$ ~~1st~~ order in Θ , 2nd order when multiplies v_b

Note: we are expanding simultaneously in $\Theta \ll 1, v \ll 1$, neglect Θv !!

$$\frac{n_e \sigma_T}{4\pi p} \int_0^\infty dp' p' \int d\Omega' \left[\delta(\vec{p}-\vec{p}') \left(-p' \frac{\partial f^{(0)}}{\partial p'} \Theta(\hat{p}') + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}) \right) \right]$$

$$+ (\vec{p}-\vec{p}') \cdot \vec{v}_b \frac{\partial \delta(\vec{p}-\vec{p}')}{\partial p'} (f^{(0)}(p') - f^{(0)}(p))$$

Only two terms depend on Ω' , need to be integrated $\int d\Omega'$

$$\propto \int d\Omega' \Theta(\hat{p}')$$

$$\propto \int d\Omega' \bar{p}' \cdot \bar{v}_b$$

Define monopole of the temperature perturbation Θ

$$\Theta_0(\bar{x}, A) \equiv \frac{1}{4\pi} \int d\Omega' \Theta(\hat{p}', \bar{x}, t)$$

(monopole) angle-averaged perturbation: CANNOT be absorbed into T since T does not depend on \bar{x} but Θ_0 still does!

Θ_0 at any point: deviation of the monopole at this point from its average across all space

$$\int d\Omega' \Theta(\hat{p}') \propto \Theta_0 \quad \int d\Omega' \bar{p}' \cdot \bar{v}_b = 0 \text{ since } \bar{v}_b \text{ fixed vector ("odd" function)}$$

$$\rightarrow C[R(\bar{p})] = \frac{n_e \sigma_T}{4\pi p} \int_0^\infty dp' p' \left[\delta(p-p') \left(-4\pi \Theta_0 p' \frac{\partial f^{(0)}}{\partial p'} + 4\pi p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}') \right) \right. \\ \left. + \frac{4\pi \bar{p} \cdot \bar{v}_b}{p'} \frac{\partial \delta(p-p')}{\partial p'} (f^{(0)}(p') - f^{(0)}(p)) \right] =$$

$$= \frac{n_e \sigma_T}{p} \int_0^\infty dp' p' \left[\delta(p-p') \left(-p \frac{\partial f^{(0)}}{\partial p'} \Theta_0 + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p}') \right) + \bar{p} \cdot \bar{v}_b \frac{\partial \delta(p-p')}{\partial p'} (f^{(0)}(p') - f^{(0)}(p)) \right] =$$

trivial: $p' \rightarrow p$ $-p^2 \frac{\partial f^{(0)}}{\partial p} (\Theta_0 - \Theta(\hat{p}'))$

By parts

$$\int_0^\infty dp' p' \frac{\partial \delta(p-p')}{\partial p'} (f^{(0)}(p') - f^{(0)}(p)) = \delta(p-p') \times (\dots) \Big|_0^\infty - \int_0^\infty dp' p' \delta(p-p') \frac{\partial f^{(0)}}{\partial p'}(p') = -p \frac{\partial f^{(0)}}{\partial p}$$

$$= \frac{n_e \sigma_T}{p} \left[-p^2 \frac{\partial f^{(0)}}{\partial p} (\Theta_0 - \Theta(\hat{p}')) - p \frac{\partial f^{(0)}}{\partial p} \bar{p} \cdot \bar{v}_b \right] = -p \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}') + \hat{p}' \cdot \bar{v}_b \right]$$

Final result for collision term

$$C[f(\vec{p})] = -p \frac{df^{(0)}}{dp} n_e \sigma_T \left[\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{v}_b \right]$$

Physical interpretation:

if $\vec{v}_b \rightarrow 0$ (electrons no bulk velocity)

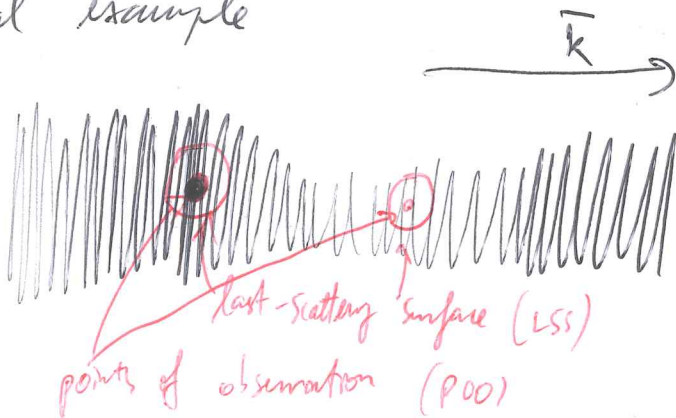
$$\Theta(\hat{p}) \rightarrow \Theta_0$$

efficient Compton scattering makes only monopole of perturbation survive because of small mean free path.

At a given point in space photons from all directions have roughly

the same temperature (uniform on sky)

Pictorial example



If electrons also have bulk velocity, photon temperature perturbation will have dipole moment (but higher order moments suppressed)

Higher order ~~moments~~ multipole moments / Legendre polynomial expansion

$$\mu \equiv \frac{\vec{k} \cdot \hat{p}}{k} = \cos \theta$$

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^1 \frac{d\mu}{2} P_l(\mu) \Theta(\mu)$$

$$P_0(\mu) = 1$$

$$P_2(\mu) = \frac{3\mu^2 - 1}{2}$$

$$P_1(\mu) = \mu$$

$$P_3(\mu) = \frac{5\mu^3 - 3\mu}{2}$$

$$\Theta(\mu) = \sum_{l=0}^{\infty} \Theta_l P_l(\mu)$$

Properties: $\int_{-1}^1 d\mu P_n(\mu) P_m(\mu) = \delta_{nm}$ if $n \neq m$

$$\frac{2}{2n+1} \int_{-1}^1 P_n(\mu) P_m(\mu) d\mu$$

$$\frac{1}{\sqrt{1-z\mu t + t^2}} = \sum_{n=0}^{\infty} P_n(\mu) t^n$$