THE BOLTZMANN EQUATIONS

Recall what we saw so far: <u>integrated</u> version of Boltzmann equation

$$\frac{1}{a^3}\frac{d(na^3)}{dt} = \int \frac{d^3p}{(2\pi)^3 2E} \cdots \delta^3(\Sigma \vec{p})\,\delta(\Sigma E)\,|M|^2 * \mathcal{F}(f)$$

$1+2\to 3+4$ $\Downarrow$ approximations

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v\rangle \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right)$$

$\Downarrow$ equilibrium, Saha equation

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}$$

$n \sim \int d^3p\, f(p)$ contains integrated information

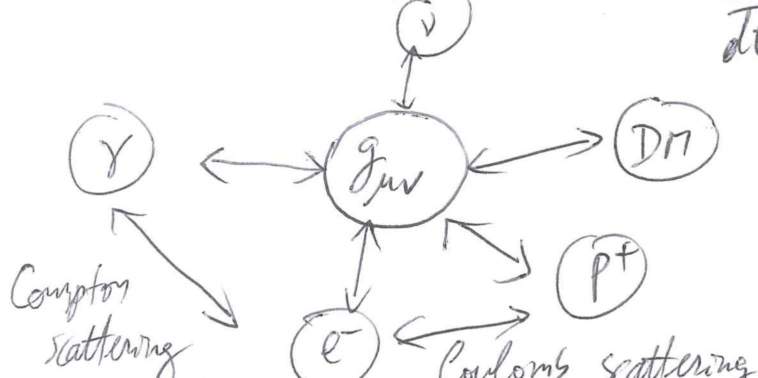
We want a more "primitive" version to get information about f

Schematically

$$\boxed{\frac{df}{dt} = C[f]}$$

collision term

In the absence of collisions $\frac{df}{dt} = 0 \longrightarrow$ number of particles in a given phase space element is constant, <u>but</u> phase space element themselves move non-trivially



Boltzmann equations for harmonic oscillator

Start from a simple example (similar to GR, simpler algebra)

$$E = \frac{p^2}{2m} + \frac{1}{2} k x^2$$

Equations of motion $\frac{dx}{dt} = \frac{p}{m}$ $\quad \frac{dp}{dt} = -kx$

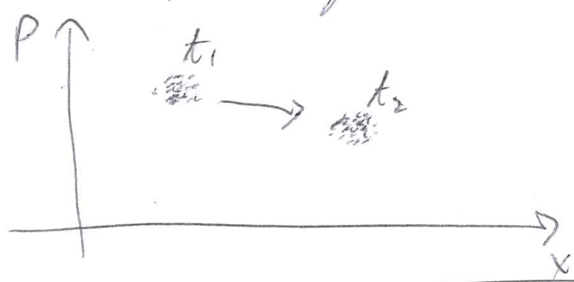
eventually will become geodesic equation

$$\frac{df}{dt} = 0$$

total derivative: not so innocuous!

$$\frac{df(t,x,p)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} = 0$$

conservation of particle number in phase space element does not mean that f "stays still" in phase space



phase space elements move keeping the same "shape", number of particles unchanged

$$\cancel{\frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x}\frac{dx}{dt}} \qquad \boxed{\frac{\partial f}{\partial t} + \underbrace{\frac{p}{m}\frac{\partial f}{\partial x}}_{\text{how quickly oscillator moves in real space}} \underbrace{- kx\frac{\partial f}{\partial p}}_{\text{how quickly particles lose momentum}} = 0}$$

Consider equilibrium distribution $\frac{\partial f}{\partial t} = 0 \longrightarrow \frac{p}{m}\frac{\partial f}{\partial x} = kx\frac{\partial f}{\partial p}$

General solution

$$f(p,x) = f_{EQ}(E)$$

f_{EQ} at equilibrium only function of energy

Check: $\frac{p}{m}\frac{\partial f(E)}{\partial x} - k x \frac{\partial f(E)}{\partial p} = \frac{p}{m}\frac{df}{dE}\frac{dE}{dx} - kx \frac{df}{dE}\frac{\partial E}{\partial p} =$

$$= \frac{df}{dE}\left[\frac{p}{m}\frac{\partial E}{\partial x} - kx \frac{\partial E}{\partial p}\right] = \frac{df}{dE}\left[\cancel{\frac{p}{m}kx - kx\frac{p}{m}}\right] = 0$$

In the presence of collisions, for there to be equilibrium these must vanish $\rightarrow f_{EQ}$ driven to equilibrium distribution, e.g. Maxwell-Boltzmann $\propto e^{-\frac{E}{T}}$

Collisionless Boltzmann equation for photons

Let's consider $\frac{df}{dt}$ for (massless) γ. Need to specify metric and perturbations around smooth Universe

Zeroth order

$$g_{\mu\nu} = \mathrm{diag}\left(-1, a^2(t), a^2(t), a^2(t)\right) \quad \text{no } \bar{x} \text{ dependence!}$$

$$g_{00}(\bar{x},t) = -1 \qquad g_{0i}(\bar{x},t) = 0 \qquad g_{ij}(\bar{x},t) = \delta_{ij}\, a^2(t)$$

Now let's perturb it: need two more functions of space and time $\Psi(\bar{x},t)$ $\Phi(\bar{x},t)$

$$g_{00}(\bar{x},t) = -1-2\Psi(\bar{x},t)$$

$$g_{0i}(\bar{x},t) = 0$$

$$g_{ij}(\bar{x},t) = a^2\delta_{ij}\left(1+2\Phi(\bar{x},t)\right)$$

Very useful to know

$$(1+x)^{\alpha} \simeq 1+\alpha x$$
$$(1-x)^{\alpha} \simeq 1-\alpha x$$

$$\boxed{ds^2 = -(1+2\Psi)dt^2 + a^2(t)\left[(1+2\Phi)\,\delta_{ij}\,dx^i dx^j\right] = a^2(\eta)\left[-(1+2\Psi)d\eta^2 + (1+2\Phi)\delta_{ij}dx^i dx^j\right]}$$

Ψ, Φ are **small**: we will drop all quadratic terms.

Sign convention: overdense region $\Psi<0$, $\Phi>0$

Ψ: Newtonian potential

Φ: perturbation to spatial curvature

Notes (technical, can gloss over for now):

- Ψ, Φ only describe **scalar** perturbations

Helmholtz decomposition

$g_{\mu\nu}$ 16 components $\xrightarrow[g_{\mu\nu}=g_{\nu\mu}]{\text{symmetric}}$ 10 independent components

most general linearized perturbation

transforms as

4 scalar

2 (divergence-free) vector

(traceless, symmetric) tensor

$4\times1+2\times2+1\times4 = 10$ dof $\longrightarrow$ 6 independent components of which 2 scalars $\rightarrow$ $\boxed{\Psi, \Phi}$

gauge invariance remove 4 components

- this is a particular choice of gauge

conformal Newtonian gauge

Alternative gauge also popular in cosmological perturbation theory

$$\boxed{ds^2 = a^2(\eta)\left[-d\eta^2+(\delta_{ij}+h_{ij})\,dx^i dx^j\right]} \longrightarrow \text{synchronous gauge}$$

We will work exclusively in conformal Newtonian gauge, so we need to figure out evolution equations for Ψ and Φ

Need to express $\frac{df}{dt}$

Idea: $\frac{d}{dt} \simeq \sum_i \frac{\partial}{\partial \alpha_i}$

$\cancel{f=f(x^\mu)}$ $f = f(\bar{x}, \bar{p}, t) = f(x^\mu, p^\mu)$ where $x^\mu = (t, \bar{x})$

note: we do NOT need to specify λ explicitly $\quad p^\mu \equiv \frac{dx^\mu}{d\lambda}$

In principle f defined on 8-dimensional space, but we have additional constraints

Photon is massless:

$$P^2 \equiv g_{\mu\nu} p^\mu p^\nu = 0 \longrightarrow p^\mu \text{ has only 3 independent components}$$

$g_{\mu\nu}$ → conformal Newtonian gauge: $\mathrm{diag}\left[-(1+2\Psi), (1+2\Phi), (1+2\Phi), (1+2\Phi)\right]$

$P^2 = 0 = -(1+2\Psi)(p^0)^2 + p^2 \cancel{=0}$ $\quad p^2 \equiv g_{ij} p^i p^j$ (only spatial indices)

$$\Rightarrow p^0 = \frac{p}{\sqrt{1+2\Psi}} \simeq p(1-\Psi) \quad \text{using } (1+x)^{-\frac{1}{2}} \simeq 1 - \frac{x}{2}$$

~~Sign convention~~ $\quad \hookrightarrow p^0 \simeq p(1-\Psi)$

~~overdense region $\Psi<0 \rightarrow 1-\Psi>1$~~

~~underdense region $\Psi>0$~~

Sign convention

overdense region $\Psi<0, \Phi>0 \Rightarrow 1-\Psi>1 \Rightarrow$ photons lose energy moving out of potential ~~well~~ well $(\Psi<0 \rightarrow 0)$

$\hookrightarrow$ redshift

underdense region $\Psi>0, \Phi<0$

$P^0 \simeq p(1-\Psi)$ basically generalizes $E=pc$ to perturbed FLRW space-time. Wherever we meet P^0 we can replace it with p ("generalized magnitude of momentum")

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial x_i}$$ → we won't include P^0, just p and its (angular) direction $\hat{p}^i = \hat{p}_i$ $\quad \delta_{ij}\hat{p}^i\hat{p}^j = 1$

unit vector

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\cdot\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\cdot\frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i}\cdot\frac{d\hat{p}^i}{dt}}$$

Strategy:
- Work out RHS to 1st order
- Expand LHS to 1st order after perturbing
- Equate 1st order terms

Next goal: figure out $\frac{dx^i}{dt}$, $\frac{dp}{dt}$, $\frac{d\hat{p}^i}{dt}$ to <u>first order</u>

Important to "see" second-order terms and set them $=0$!!!

$\cancel{\frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}} \to 0$ to 1st order

$f^{(0)}$: Bose-Einstein $\quad f^{(0)} = f^{(0)}(p) \quad \frac{\partial f^{(0)}}{\partial \hat{p}} = 0$ no dependence on direction!

So $\frac{\partial f}{\partial \hat{p}^i}$ necessarily (at least) 1st order,
$\frac{d\hat{p}^i}{dt}$ also (at least) 1st order
} $\frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}$ (at least) second order!

↳ $$\frac{df}{dt} \simeq \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt}$$

Let's move on to $\frac{\partial f}{\partial x^i}\frac{dx^i}{dt}$

we $P^\mu \equiv \frac{dx^\mu}{d\lambda}$

$$p^\mu \equiv \frac{dx^\mu}{d\lambda} \longrightarrow p^i \equiv \frac{dx^i}{d\lambda}, \quad p^0 \equiv \frac{dt}{d\lambda}$$

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{dx^i}{d\lambda}\left(\frac{dt}{d\lambda}\right)^{-1} = \frac{p^i}{p^0}$$

Let's use p, $\hat{p}^i$

$$p^0 \simeq p(1-\Psi) \qquad p^i \equiv C\hat{p}^i \rightarrow C=?$$

$$p^2 = g_{ij}p^ip^j = g_{ij}\hat{p}^i\hat{p}^jC^2 = a^2(1+2\Phi)\underbrace{\delta_{ij}\hat{p}^i\hat{p}^j}_{=1}C^2 = a^2(1+2\Phi)C^2$$

$$p^2 = a^2(1+2\Phi)C^2 \rightarrow p = a(1+2\Phi)^{1/2}C \rightarrow C = \frac{p}{a}(1+2\Phi)^{-1/2} \simeq \frac{p(1-\Phi)}{a}$$

$$p^i = C\hat{p}^i = p\hat{p}^i\frac{1-\Phi}{a}$$

$$\frac{dx^i}{dt} = \frac{p^i}{p^0} = p\hat{p}^i\frac{1-\Phi}{a}\left[p(1-\Psi)\right]^{-1} \simeq \frac{p}{p}\frac{\hat{p}^i}{a}(1-\Phi)(1+\Psi) \simeq \frac{\hat{p}^i}{a}(1+\Psi-\Phi)$$

$$(1-\Psi)^{-1} \simeq (1+\Psi)$$

$$(1-\Phi)(1+\Psi) = 1+\Psi-\Phi-\underbrace{\cancel{\Psi\Phi}}_{\text{2nd order}} \simeq 1+\Psi-\Phi$$

$$\frac{dx^i}{dt} \simeq \frac{\hat{p}^i}{a}(1+\Psi-\Phi)$$

"v_γ" photon velocity

Does it make sense?

Recall overdensity $\Psi<0$, $\Phi>0$

$1+\Psi-\Phi<1 \rightarrow$ photons slow down in an overdensity

$$\underbrace{\frac{\partial f}{\partial x^i}}_{\text{1st order}}\underbrace{\frac{dx^i}{dt}}_{\text{keep only 0th order}} \qquad \frac{\partial f^{(0)}}{\partial x^i} = 0 \text{ so } \frac{\partial f}{\partial x^i} \neq 0 \text{ only at 1st order}$$

keep only 0th order $\rightarrow$ neglect potentials

$$\frac{\partial f}{\partial x^i}\frac{dx^i}{dt} \underset{\text{1st order}}{\simeq} \frac{\partial f}{\partial x^i}\frac{\hat{p}^i}{a}$$

What do we have so far?

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underbrace{\frac{\partial f}{\partial x^i}\frac{\hat{p}^i}{a}}_{\simeq \frac{\partial f}{\partial x^i}\frac{dx^i}{dt}} + \frac{\partial f}{\partial p}\frac{dp}{dt} + \underbrace{\cancel{\frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}}}_{\text{second order (at least)}} \to 0$$

We need $\frac{dp}{dt} \to$ geodesic equation!

$$\frac{dp^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = -\Gamma^\mu_{\alpha\beta}p^\alpha p^\beta$$

$$\frac{dp^0}{d\lambda} = -\Gamma^0_{\alpha\beta}p^\alpha p^\beta \qquad \left\{ \frac{dp^0}{d\lambda} = \frac{dp^0}{dt}\frac{dt}{d\lambda} = p^0\frac{dp^0}{dt} \simeq p(1-\psi)\frac{d}{dt}[p(1-\psi)] \right.$$

$$p^0 = \frac{dt}{d\lambda}, \quad p^0 \simeq p(1-\psi)$$

$$p(1-\psi)\frac{d}{dt}[p(1-\psi)] = -\Gamma^0_{\alpha\beta}p^\alpha p^\beta \Longrightarrow \frac{d}{dt}[p(1-\psi)] \simeq -\frac{\Gamma^0_{\alpha\beta}p^\alpha p^\beta}{p}(1+\psi)$$

$$\frac{1}{p(1-\psi)} \simeq \frac{1+\psi}{p}$$

$$\left(\frac{d}{dt}[p(1-\psi)] = (1-\psi)\frac{dp}{dt} - p\frac{d\psi}{dt}\right)$$

$$\Longrightarrow (1+\psi)\times\frac{dp}{dt}(1-\psi) = \left[p\frac{d\psi}{dt} - \frac{\Gamma^0_{\alpha\beta}p^\alpha p^\beta}{p}(1+\psi)\right]\times(1+\psi)$$

$\xrightarrow{\text{1st order}}$

$(1+\psi)(1-\psi) \simeq 1$

$(1+\psi)^2 \simeq 1+2\psi$

$\frac{d\psi}{dt}(1+\psi) \simeq \frac{d\psi}{dt}$

$$\frac{dp}{dt} = p\frac{d\psi(\vec{x},t)}{dt} - \frac{\Gamma^0_{\alpha\beta}p^\alpha p^\beta}{p}(1+2\psi) =$$

$$= p\underbrace{\left[\frac{\partial\psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\psi}{\partial x^i}\right]}_{\frac{dx^i}{dt}\frac{\partial\psi}{\partial x^i}} - \frac{\Gamma^0_{\alpha\beta}p^\alpha p^\beta}{p}(1+2\psi)$$