RECAP OF GR AND THE SMOOTH UNIVERSE

(~ Dodelson Chapter 2)

- FLRW metric
$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix}$$

Geodesic equation
$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

Christoffel symbol
$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right]$$

Einstein equations
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \equiv G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Perfect fluid
$$\cancel{T_{\mu\nu} = \mathrm{diag}} \quad T^\mu{}_\nu = \mathrm{diag}(-\rho, P, P, P)$$

Energy evolution
$$\nabla_\mu T^\mu{}_\nu = 0 \implies \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$$

Hubble rate
$$H \equiv \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt} \longrightarrow \dot{\rho} + 3H(\rho + P) = 0$$

Equation of state
$$w \equiv \frac{P}{\rho} \begin{cases} 0 & \text{matter ("dust")} \\ \frac{1}{3} & \text{radiation} \\ -1 & \Lambda \end{cases} \implies \dot{\rho} + 3H\rho(1+w) = 0$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w) = 0 \longrightarrow \frac{d\rho}{\cancel{dt}} + 3\frac{1}{a}\frac{da}{\cancel{dt}}\rho(1+w) = 0 \longrightarrow \int\frac{d\rho}{\rho} = -3(1+w)\int\frac{da}{a}$$

$$\ln\rho = \ln\left[a^{-3(1+w)}\right] + C \longrightarrow \rho = \rho_0 a^{-3(1+w)} \begin{cases} \propto a^{-3} & \text{matter} \\ \propto a^{-4} & \text{radiation } (E \propto \frac{1}{a}) \\ \text{const} & \Lambda \end{cases}$$

Distribution function $dN = f(\bar{x}, \bar{p})\, d^3\bar{x}\, d^3\bar{p}$

- $$\rho_i \equiv g_i \int \frac{d^3p}{(2\pi)^3} E(\bar{p}) f_i(\bar{x}, \bar{p}) \qquad P_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\bar{x}, \bar{p}) \frac{p^2}{3E(\bar{p})}$$

$$s \equiv \frac{\rho + P}{T} \quad \text{entropy density} \propto a^{-3} \longrightarrow s a^3 = \text{const}$$

~~Distances~~ Friedmann equations

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$\frac{1}{a}\frac{d^2a}{dt^2} + \frac{1}{2}\left(\frac{1}{a}\frac{da}{dt}\right)^2 = -4\pi G P \quad \left[\text{or equivalently } \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p)\right]$$

First Friedmann equation

$$\rho_{cr} \equiv \frac{3H_0^2}{8\pi G} \quad \text{critical energy density} \qquad \Omega_i = \frac{\rho_i(a)}{\rho_{cr}}$$

$$H^2 = H_0^2 \sum_i \Omega_i(a) = H_0^2\left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_k a^{-2} + \dots\right]$$

$$\rho = \rho_m a^{-3} + \rho_r a^{-4} + \rho_\Lambda + \rho_k a^{-2} + \dots$$

Distances

Comoving distance / comoving time $\quad d\eta = \frac{dt}{a(t)} \to \eta = \int_0^t \frac{dt'}{a(t')}$

$\frac{d}{d\eta} = a\frac{d}{dt}$

Note: $\eta = \int_0^t \frac{dt'}{a(t')} = \int_\infty^z \frac{dt'}{da'}\frac{da'}{dz'}\frac{dz'}{a(z')} =$ $\quad \left[\frac{da}{dz} = \frac{d}{dz}\left(\frac{1}{1+z}\right) = -\frac{1}{(1+z)^2}\right.$

$$= -\int_\infty^z \frac{1}{\dot a}\frac{1}{(1+z)^2}\frac{dz'}{a(z')} = -\int_\infty^z \frac{1}{\dot a}\, a \frac{1}{\cancel{a(z')(1+z')}}\, dz' =$$

→ Comoving Hubble rate is more useful!

$$= -\int_\infty^z \left(\frac{\dot a}{a}\right)^{-1} dz' = -\int_\infty^z \frac{dz'}{H(z')} = \int_z^\infty \frac{dz'}{H(z')}$$

$$\mathcal{H} = \frac{1}{a}\frac{da}{d\eta} = \frac{1}{a}\frac{da}{dt}a = aH \to H = \frac{\mathcal{H}}{a}$$

Angular diameter distance (flat) $\quad d_A = \frac{\eta}{1+z}$

Luminosity distance (flat) $\quad d_L = (1+z)\eta$

$$d_L = (1+z)^2 d_A$$

↓

Etherington distance-duality relation

Cosmic inventory

- Photons

$$\rho_\gamma = \frac{\pi^2}{15} T_\gamma^4 \qquad \rho_\gamma(a) = \frac{\pi^2}{15} T_\gamma(a)^4 \propto a^{-4}$$

T_γ measured by CMB $\quad T_\gamma \sim 2.73K$

$$\Omega_r = \frac{\rho_\gamma}{\rho_{cr}} \sim 10^{-5} \qquad (\text{useful: } 10^4 K \sim 1\, eV)$$

- "Baryons" (visible matter)

$$\rho_b(a) = \Omega_b\, a^{-3}$$

$\hookrightarrow$ measured $\Omega_b \sim 0.05$

- Matter (baryons + DM) $\qquad$ Matter-radiation equality $z_{eq} \sim 3000$

$$\rho_m(a) = \Omega_m\, a^{-3}$$

$\hookrightarrow$ measured $\Omega_m \sim 0.3 \Longrightarrow \Omega_c \sim 0.25$

$\hookrightarrow$ "c" = "cold" (dark matter)

- Neutrinos

Using $sa^3 = \text{const} \longrightarrow T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$

$$\longrightarrow \rho_\nu = \frac{21}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma \qquad (\text{massless neutrinos})$$

$$\cancel{\rho_\nu = \frac{\sum m_\nu}{94}} \qquad \Omega_\nu \simeq \frac{\sum m_\nu}{57\, eV} \qquad (\text{massive neutrinos})$$

- Dark energy

$$\rho = \text{const} = \frac{\Lambda}{8\pi G} \qquad \Omega_\Lambda \sim 0.7$$

More general DE component $\quad \rho(a) = \rho_0\, a^{-3(1+w)} \qquad w < -\frac{1}{3}$