Cosmological Tensions Lecture 1
Basics of theoretical and observational cosmology

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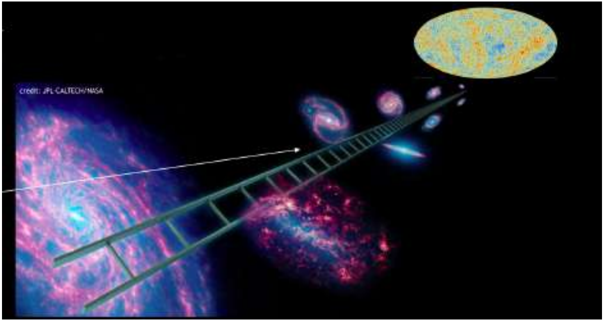
Why care about H_0?

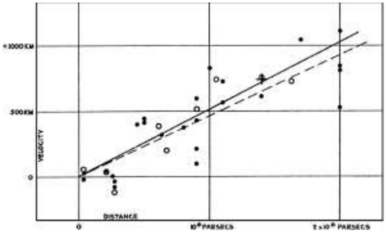
- Allan Sandage, 1970: "*Cosmology [is] the search for two numbers: the current rate [H_0] and deceleration of the expansion [q_0]*"
- Adam Riess, 2019: "*H_0 is the ultimate end-to-end test for ΛCDM*"



Credits: Credits: JPL-Caltech/NASA and Dillon Brout

The expanding Universe

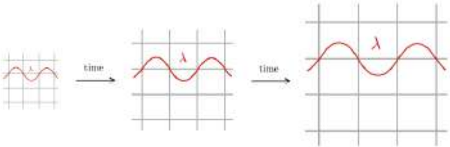
The first Hubble diagram



Hubble, PNAS 15 (1929) 168

$$v = Hd$$

Redshift, comoving grid, scale factor, Hubble rate



Credits: Marco Scalisi

$$\begin{aligned}
\lambda_{\rm obs} &= \lambda_{\rm emit}(1+z) \\
\frac{1}{a} &= 1+z \qquad (a_0 \equiv 1) \\
H(t) &\equiv \frac{1}{a}\frac{da}{dt} \\
H_0 &\equiv 100h\,\mathrm{km/s/Mpc}
\end{aligned}$$

General Relativity and the FLRW metric

Einstein equations:

$$G_{\mu\nu}(+\Lambda g_{\mu\nu}) \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R(+\Lambda g_{\mu\nu}) = 8\pi G T_{\mu\nu}$$

Energy-momentum tensor for a perfect fluid:

$$T^{\mu}_{\nu} = \mathrm{diag}\left(-\rho, P, P, P\right)$$

Ricci tensor and Ricci scalar:

$$R_{\mu\nu} \equiv \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}\,, \quad R \equiv g^{\mu\nu} R_{\mu\nu}$$

Christoffel symbols (**NOT** a tensor):

$$\Gamma^\mu_{\alpha\beta} \equiv \frac{g^{\mu\nu}}{2}\left(\partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\beta\nu} - \partial_\nu g_{\alpha\beta}\right)$$

Friedmann-Lemaître-Robertson-Walker (FLRW) metric (with $c = 1$):

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]\,, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Friedmann and continuity equations

First Friedmann equation:

$$H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho_{\text{tot}} - \frac{k}{a^2}\,, \quad \rho_{\text{tot}} = \rho + \rho_\Lambda = \rho + \frac{\Lambda}{8\pi G}$$

Critical density and fractional density parameters:

$$\rho_{\text{crit}} \equiv \frac{3H_0^2}{8\pi G}\,, \quad \Omega_i \equiv \frac{\rho_{i,0}}{\rho_{\text{crit}}}\,, \quad \Omega_K \equiv -\frac{k}{H_0^2} \implies \sum_i \Omega_i = 1$$

Second Friedmann equation (acceleration equation):

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3P)$$

Bianchi identity and continuity equation:

$$\nabla_\mu G^{\mu\nu} = 0 \implies \nabla_\mu T^{\mu\nu} = 0 \implies \frac{d\rho}{dt} + 3H(\rho + P) = 0$$

Equation of state and solution to continuity equation:

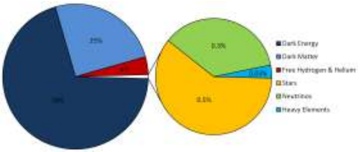
$$w \equiv \frac{P}{\rho} \implies \rho(a) = \rho_0 a^{-3(1+w)}$$

Cosmic inventory

- (Non-relativistic) **Matter**: $w \approx 0 \implies \rho_m \propto a^{-3}\,, a(t) \propto t^{\frac{2}{3}}$
  $\Omega_m \sim 0.3\,, \quad \rho_m = \Omega_m \rho_{\rm crit}\, a^{-3}$
  - **(Cold) Dark matter**: $\Omega_c \sim 0.25\,, \quad \rho_c = \Omega_c \rho_{\rm crit}\, a^{-3}$
  - **Baryons**: $\Omega_b \sim 0.05\,, \quad \rho_b = \Omega_b \rho_{\rm crit}\, a^{-3}$
- **Photons**: $w = 1/3 \implies \rho_\gamma \propto a^{-4}\,, a(t) \propto t^{\frac{1}{2}}$
  $\Omega_\gamma \sim 5 \times 10^{-5}\,, \quad \rho_\gamma(a) = \pi^2/15\, T_\gamma(a)^4\,, \quad T_\gamma(a) = 2.73\,\mathrm{K}/a(t)$
- **(Massive) neutrinos**: transition from relativistic to non-relativistic
  $\sum_i m_{\nu,i} \lesssim \mathcal{O}(0.1)\,\mathrm{eV} \implies \Omega_\nu \lesssim 2 \times 10^{-3}$
- **Dark energy**: if it is a cosmological constant, $w = -1 \implies \rho_\Lambda = \text{const}\,, a(t) \propto e^t$
  $\Omega_\Lambda \sim 0.7\,, \quad \rho_\Lambda = \Omega_\Lambda \rho_{\rm crit}$
  More generally can have $w \neq -1\,, \quad \rho_{\rm DE} = \Omega_{\rm DE} \rho_{\rm crit}\, a^{-3(1+w)}$ (as long as $w < -1/3$)

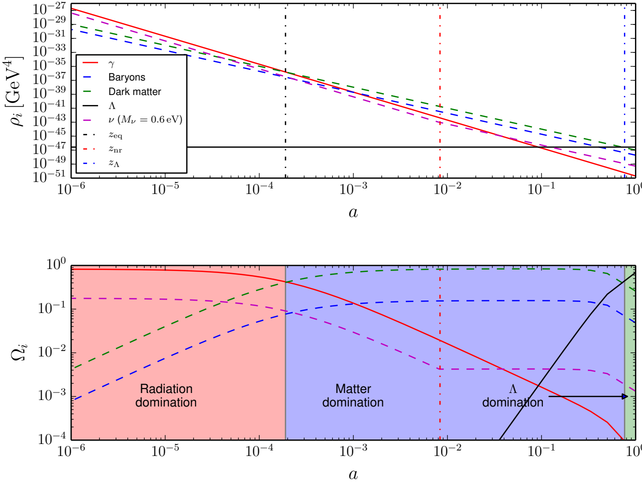
First Friedmann equation revisited:

$$H(z) = H_0 \sqrt{\Omega_m (1+z)^3 + \Omega_\gamma (1+z)^4 + \Omega_{\rm DE} (1+z)^{3(1+w)} + \Omega_\nu(z) + \Omega_K (1+z)^2}$$



Credits: Andrew Colvin

Time evolution of the cosmic inventory



My PhD thesis, arXiv:1907.08010

Distances and horizons

Conformal/comoving time/horizon/distance:

$$d\eta \equiv \frac{dt}{a(t)} \implies \eta = \int_0^t \frac{dt'}{a(t')} = \int_z^\infty \frac{dz'}{H(z')}\,, \quad \chi(z) = \int_0^z \frac{dz'}{H(z')}$$

Comoving Hubble radius:

$$\chi_H \equiv \frac{1}{aH}$$

> Two objects cannot communicate *today* if they are separated by $d > \chi_H$, but could *never* have communicated if they are separated by $d > \eta$

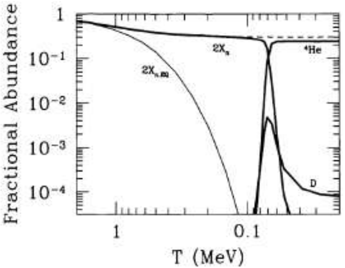
Luminosity and angular diameter distances (in a spatially flat Universe):

$$d_A = \frac{s}{\theta} \implies d_A(z) = \frac{\chi(z)}{1+z} \qquad\qquad F = \frac{L}{4\pi d_L^2} \implies d_L(z) = (1+z)\chi(z)$$

$d_L(z) = (1+z)^2 d_A(z)$ (Etherington distance-duality relation)

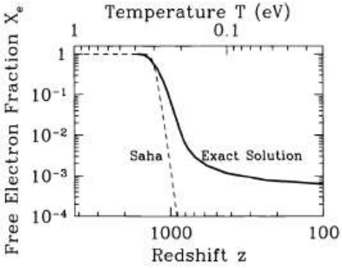
Things get interesting out of equilibrium...

$T \sim 0.01\,\mathrm{MeV}$: BBN

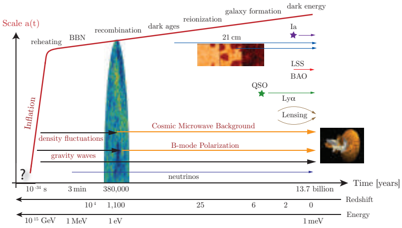


Credits: Scott Dodelson

$T \sim 0.1\,\mathrm{eV}$: recombination



Credits: Scott Dodelson



Credits: Daniel Baumann

Coupled Einstein-Boltzmann equations

- how (perturbations to) the metric affect (perturbations to) particle distributions $\rightarrow$ *(perturbed) Boltzmann equations*
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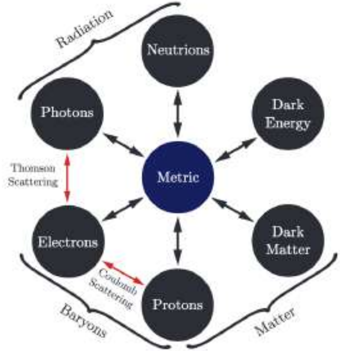


Credits: Hans Winther

Equations to perturb:

$$
\begin{aligned}
G_{\mu\nu} &= 8\pi G T_{\mu\nu} \implies \delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \\
\frac{df}{dt} &= C[f] \implies \delta\left(\frac{d(f+\delta f)}{dt}\right) = \delta C[f+\delta f]
\end{aligned}
$$

Perturbations

Scalar perturbations to the metric (conformal Newtonian gauge):

$$g_{\mu\nu} = \mathrm{diag}\left[-1 - 2\Psi(\mathbf{x}, t), a^2(t)\delta_{ij}\left(1 + 2\Phi(\mathbf{x}, t)\right)\right]$$

Perturbed photon distribution (analogously for massless νs with $\Theta \to \mathcal{N}$):

$$f(\mathbf{x}, p, t) = \left\{\exp\left[\frac{p}{T(t)\left(1 + \Theta(\mathbf{x}, p, t)\right)}\right] - 1\right\}^{-1}$$

Photon temperature perturbation multipole moments:

$$\Theta_\ell \equiv \frac{1}{(-i)^\ell}\int_{-1}^{1}\frac{d\mu}{2}\,\mathcal{P}_\ell(\mu)\Theta(\mu)\,, \qquad \mu \equiv \cos(\mathbf{k}, \hat{p})$$

Dark matter first two moments (analogously for baryons with $\delta \to \delta_b$, $v \to v_b$):

$$n_{\mathrm{dm}} = \int \frac{d^3p}{(2\pi)^3}\, f_{\mathrm{dm}} \equiv n_{\mathrm{dm}}^{(0)}(t)\left[1 + \delta(\mathbf{x}, t)\right]\,, \qquad v^i \equiv \frac{1}{n_{\mathrm{dm}}}\int \frac{d^3p}{(2\pi)^3}\, f_{\mathrm{dm}}\frac{p\hat{p}^i}{E(p)}$$

Coupled Einstein-Boltzmann system

Same set of coupled ODEs *for each k* (in the linear regime)

$$
\begin{cases}
\dot\Theta + ik\mu\Theta = -\dot\Phi - ik\mu\Psi - \dot\tau\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}\mathcal{P}_2(\mu)\Pi\right] & [\dot\tau \equiv -n_e\sigma_T a] \\
\Pi = \Theta_2 + \Theta_{P2} + \Theta_{P0} \\
\dot\Theta_P + ik\mu\Theta_P = -\dot\tau\left[-\Theta_P + \frac{1}{2}\left(1 - \mathcal{P}_2(\mu)\right)\right] \\
\dot\delta + ikv = -3\dot\Phi \\
\dot v + \frac{\dot a}{a}v = -ik\Psi \\
\dot\delta_b + ikv_b = -3\dot\Phi \\
\dot v_b + \frac{\dot a}{a}v_b = -ik\Psi + \frac{\dot\tau}{R}\left(v_b + 3i\Theta_1\right) & [R \equiv 3\rho_b/4\rho_\gamma] \\
\dot{\mathcal{N}} + ik\mu\mathcal{N} = -\dot\Phi - ik\mu\Psi \\
k^2\Phi + 3\frac{\dot a}{a}\left(\dot\Phi - \frac{\dot a}{a}\Psi\right) = 4\pi G a^2\left(\rho_c\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0\right) \\
k^2\left(\Phi + \Psi\right) = -32\pi G a^2\left(\rho_\gamma\Theta_2 + \rho_\nu\mathcal{N}_2\right) \\
\ddot h_\alpha + 2\frac{\dot a}{a}\dot h_\alpha + k^2 h_\alpha = 0 & [\alpha = +, x]
\end{cases}
$$

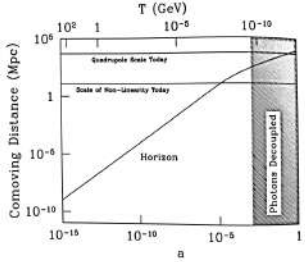
Initial conditions:

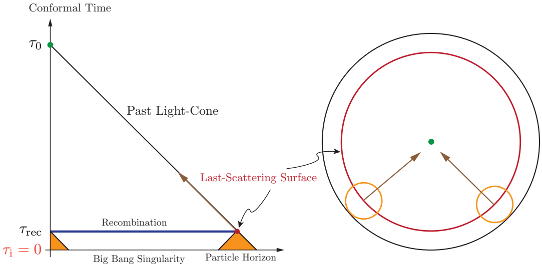
$$
\begin{cases}
\Phi(k,\eta_i) = -\Psi(k,\eta_i) = 2\Theta_0(k,\eta_i) = 2\mathcal{N}_0(k,\eta_i) = \Phi_p(k) \\
\delta(k,\eta_i) = \delta_b(k,\eta_i) = \frac{3}{2}\Phi_p(k) \\
\Theta_1(k,\eta_i) = \mathcal{N}_1(k,\eta_i) = \frac{iv(k,\eta_i)}{3} = \frac{iv_b(k,\eta_i)}{3} = -\frac{k\Phi_p}{6aH}
\end{cases}
$$

Note: $4\Theta_0$ "$\sim$" δ_γ, $4\mathcal{N}_0$ "$\sim$" δ_ν, $-3i\Theta_1$ "$\sim$" v_γ, $-3i\mathcal{N}_1$ "$\sim$" v_ν

Comoving wavelengths versus comoving horizon

At any given time there is a mode of increasingly large wavelength *entering* the horizon





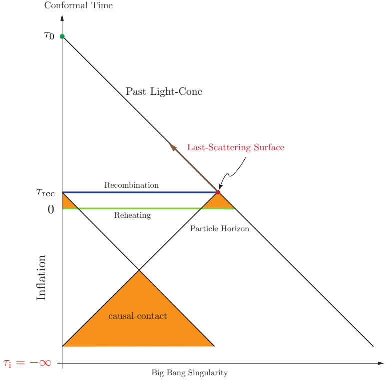
Credits: Scott Dodelson (left); Daniel Baumann (right)

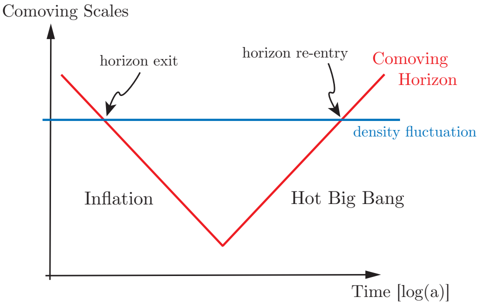
Horizon problem: why is the CMB so uniform even on the largest scales?

Flatness problem: why is the Universe so close to being spatially flat ($\Omega_K = 0$ is an unstable fixed point in FLRW)?

Inflation

Period of accelerated expansion in the early Universe makes $1/aH$ shrink, η picks up most of its contributions at early times, and Universe is naturally "flattened"





Credits: Daniel Baumann

Outcomes of inflation

Most of inflation's expected outcomes are seen in data:†

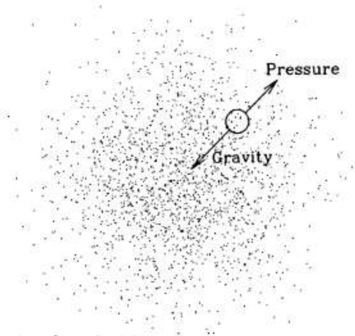
- Close to spatially flat Universe $\checkmark$
- Nearly scale-invariant fluctuations $P_\Phi(k) \sim k^{-3}$ $\checkmark^\star$
- Nearly Gaussian fluctuations $\checkmark$
- Mostly adiabatic fluctuations $\checkmark$
- Phase coherence (inflation excites "cosine mode") $\checkmark$
- Coherent superhorizon fluctuations (especially $\ell < 100$ TE) $\checkmark$
- Primordial, nearly scale-invariant, nearly Gaussian tensor modes ?

$^\star\Delta^2(k) \propto k^3 P(k) = k^{n_s - 1}$, with $n_s \sim 0.96$

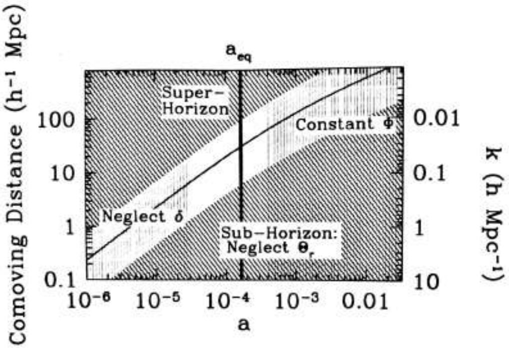
†This is not to say inflation doesn't have problems, some would actually say quite the opposite...

Understanding inhomogeneities

We cannot predict exact *realization* of inhomogeneities, but we can predict their statistics $\implies$ *Goal*: solve for Φ and δ, ultimately care about $P_\delta(k)$



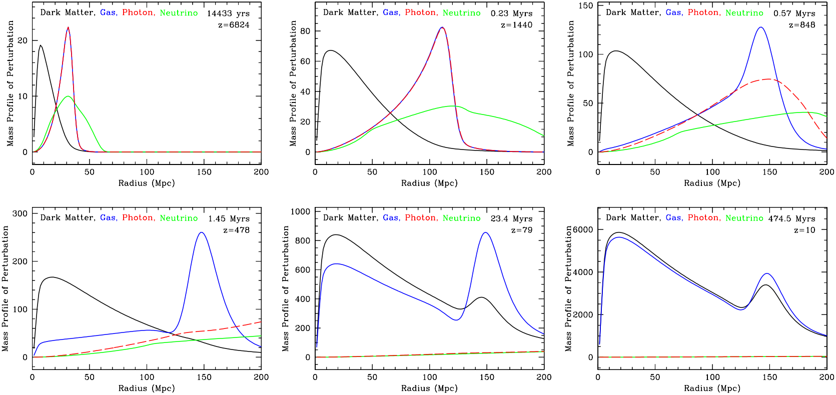
Credits: Scott Dodelson



$$\Phi(k,a) = \frac{9}{10}\Phi_p(k)T(k)\frac{D_1(a)}{a} \implies \delta(k,a) \propto k^2\Phi_p(k)T(k)D_1(a) \implies P_\delta(k) \propto k^4T^2(k)P_\Phi(k)$$

Baryon Acoustic Oscillations

$$r_s = \int_0^{t_\star} dt\, \frac{c_s(t)}{a(t)} = \int_{z_\star}^{\infty} dz\, \frac{c_s(z)}{H(z)} \simeq \mathcal{O}(150)\,\mathrm{Mpc}$$

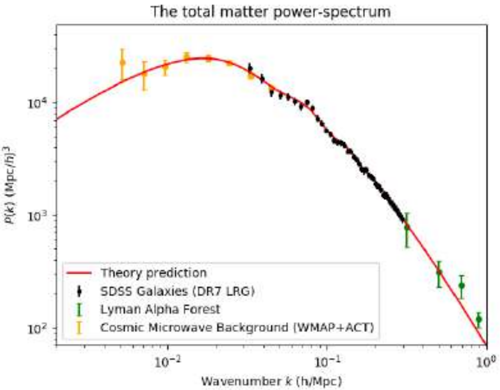


Eisenstein, Seo & White, ApJ 664 (2007) 660

Matter power spectrum

Important features:

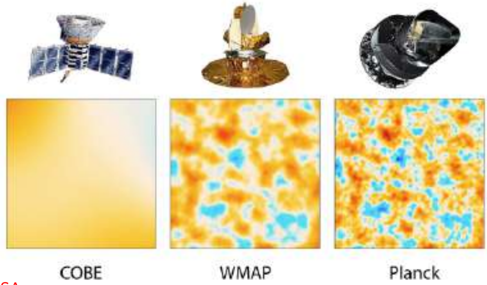
- Equality turn-around
- BAOs
- Overall normalization $\propto 1/\Omega_m$



Credits: Hans Winther

Understanding anisotropies

We cannot predict exact *realization* of anisotropies, but we can predict their statistics $\implies$ *Goal*: solve for Θ, ultimately care about C_ℓ



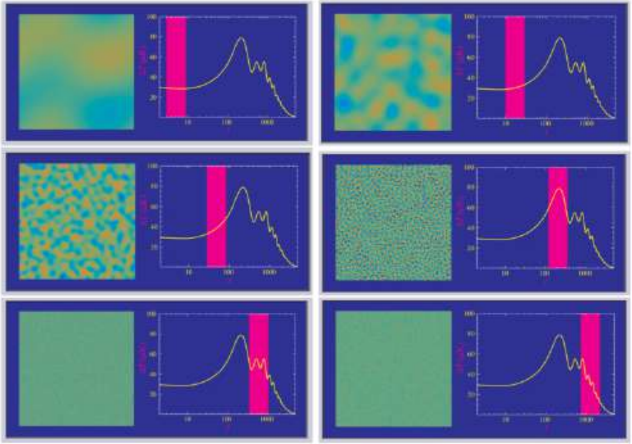
Credits: NASA/JPL-Caltech/ESA

$$\Theta(\hat{n}) = \sum_\ell \sum_m a_{\ell m} Y_{\ell m}(\hat{n}) \implies \langle a_{\ell m} \rangle = 0\,, \quad \langle a_{\ell m} a^\star_{\ell' m'} \rangle = \delta_{\ell m} \delta_{\ell' m'} C_\ell$$

Physical meaning of the CMB power spectrum

$\ell \sim \pi/\theta$: inverse angular scale

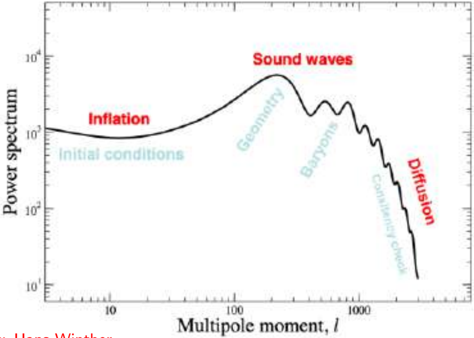
C_ℓ: indication of how much T fluctuates with respect to the average in patches associated with given angular size



Credits: Wayne Hu

CMB temperature anisotropy power spectrum

Credits: Tonale Winter school organizers



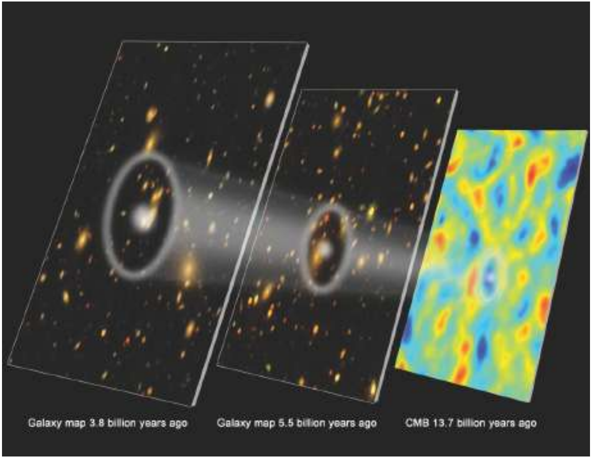
Credits: Hans Winther

Physical effects:

- SW
- Acoustic oscillations
- Damping
- Even-odd peak modulation (baryons)
- Secondary anisotropies (ISW, lensing)

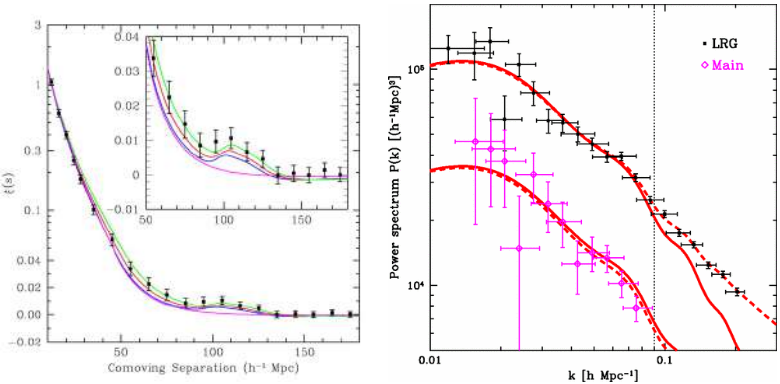
Baryon Acoustic Oscillations beyond the CMB

BAO feature can also be detected statistically in the late-time clustering of large-scale structure tracers



Credits: Eric Huff and the BOSS/SPT collaborations

What can we measure from BAO?



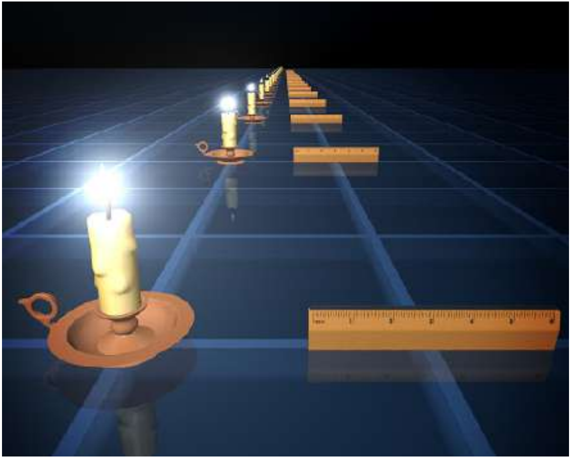
Eisenstein *et al.*, ApJ 633 (2005) 560 (left); Tegmark *et al.*, PRD 74 (2006) 123507 (right)

$$\theta_{\mathrm{BAO}}(z_{\mathrm{eff}}) = \frac{r_s}{d(z_{\mathrm{eff}})}$$

$$d(z_{\mathrm{eff}}) = \left\{ d_A(z_{\mathrm{eff}}), d_H(z_{\mathrm{eff}}) = \frac{c}{H(z_{\mathrm{eff}})}, d_V(z_{\mathrm{eff}}) = \left[(1+z_{\mathrm{eff}})^2 d_A^2(z_{\mathrm{eff}}) \frac{c z_{\mathrm{eff}}}{H(z_{\mathrm{eff}})}\right]^{\frac{1}{3}} \right\}$$

Standard rulers and standard candles

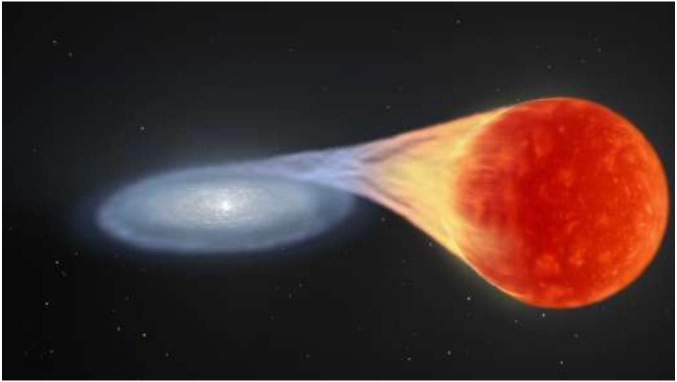
BAO are an example of standard ruler: once calibrated through r_s, they are an *absolute* distance (and therefore expansion rate) indicator



Credits: NASA/JPL-Caltech/R. Hurt (SSC)

Type Ia Supernovae as standard candles

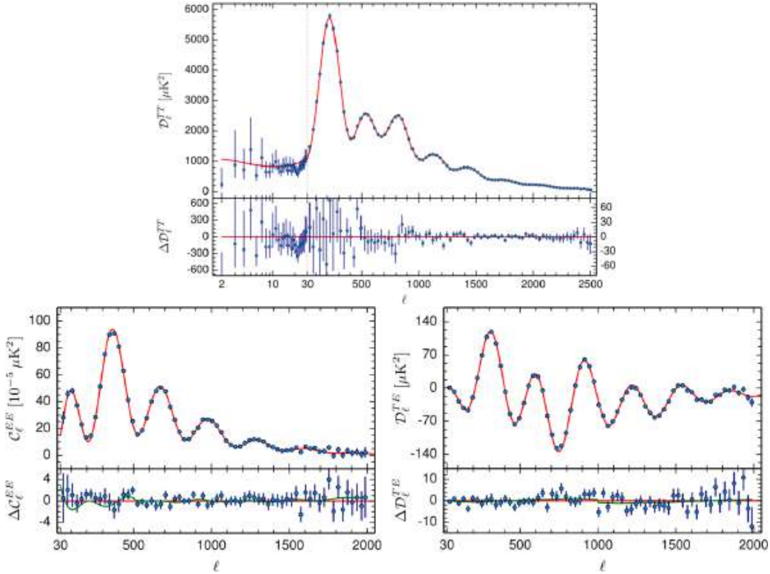
$$\mu(z) = 5\log_{10}\left[\frac{d_L(z)}{10\,\mathrm{pc}}\right] = m_B - M_B$$



Credits: NASA, Adriana Manrique Gutierrez, Aaron E. Lepsch & Scott Wiessinger

If M_B is not known, high-z (cosmographic) SNeIa are a *relative* distance indicator sensitive to unnormalized expansion rate $E(z) \equiv H(z)/H_0$

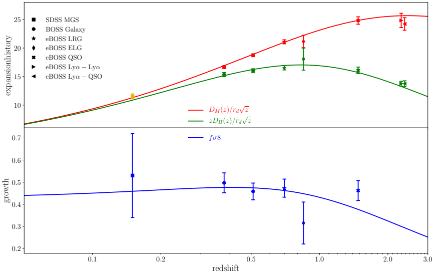
Planck power spectra

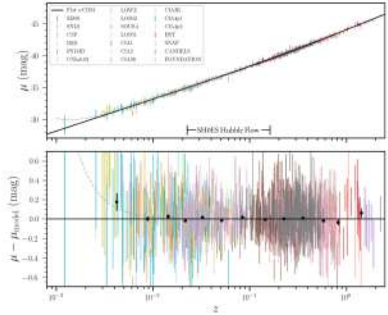


Planck collaboration, A&A 641 (2020) A6

Current BAO and cosmographic SNeIa measurements

State-of-the-art: eBOSS BAO measurements and PantheonPlus sample of cosmographic SNeIa (with or without distance ladder calibration from SH0ES, more in Lecture 2)





eBOSS collaboration, PRD 103 (2021) 083533 (left); Scolnic *et al.*, ApJ 938 (2022) 110 (right)

The ΛCDM model

Von Neumann revisited: "*With 4 parameters I can fit an elephant, with 5 I can make him wiggle his trunk, and with 6 I can fit Planck data*"

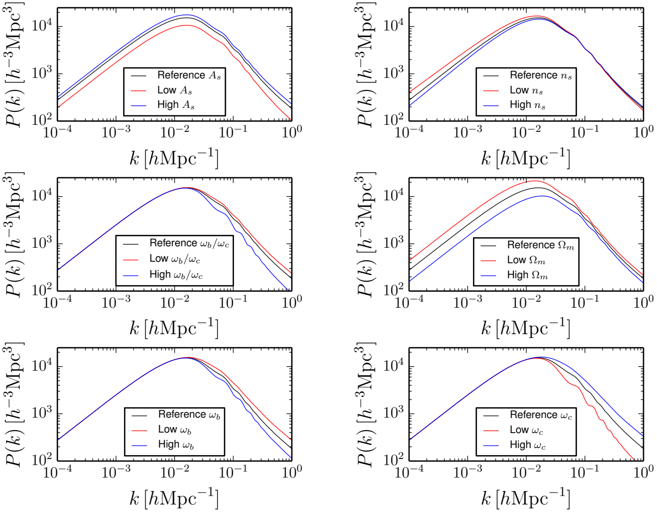
- $\omega_b = \Omega_b h^2$
- $\omega_c = \Omega_c h^2$
- $\theta_s = r_s / d_A(z_\star)$
- τ
- A_s
- n_s

Other parameters: $w = -1$, $w_a = 0$, $\Omega_K = 0$, $N_{\rm eff} = 3.044$, $M_\nu = 0.06\,\mathrm{eV}$, $\alpha_s \equiv dn_s/d\ln k = 0$, $\beta_s \equiv d^2 n_s/d(\ln k)^2 = 0$, $A_{\rm lens} = 1$, $Y_p = Y_p(\omega_b)$

Parameter	Plik best fit	Plik [1]	CamSpec [2]	$([2]-[1])/\sigma_1$	Combined
$\Omega_b h^2$	0.022383	0.02237 ± 0.00015	0.02229 ± 0.00015	-0.5	0.02233 ± 0.00015
$\Omega_c h^2$	0.12011	0.1200 ± 0.0012	0.1197 ± 0.0012	-0.3	0.1198 ± 0.0012
$100\theta_{MC}$	1.040909	1.04092 ± 0.00031	1.04087 ± 0.00031	-0.2	1.04089 ± 0.00031
τ	0.0543	0.0544 ± 0.0073	$0.0536^{+0.0069}_{-0.0077}$	-0.1	0.0540 ± 0.0074
$\ln(10^{10}A_s)$	3.0448	3.044 ± 0.014	3.041 ± 0.015	-0.3	3.043 ± 0.014
n_s	0.96605	0.9649 ± 0.0042	0.9656 ± 0.0042	$+0.2$	0.9652 ± 0.0042

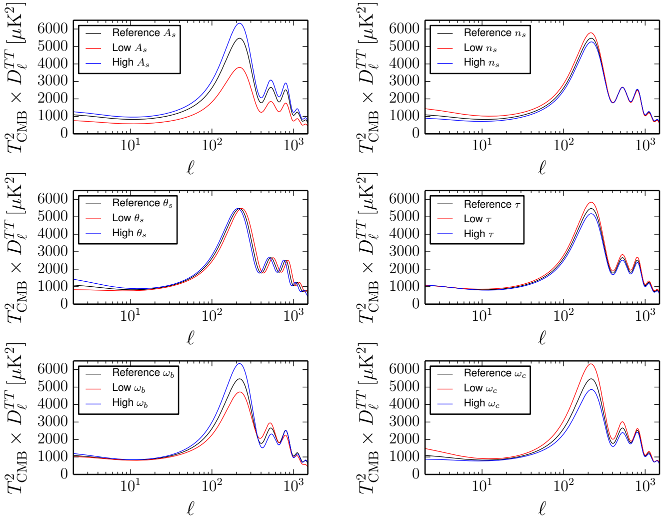
Planck collaboration, A&A 641 (2020) A6

Impact of cosmological parameters on $P(k)$



My PhD thesis, arXiv:1907.08010

Impact of cosmological parameters on C_ℓ



My PhD thesis, arXiv:1907.08010

Geometrical degeneracy

Geometrical degeneracy (CMB observations alone)

How far away? d
How tall? h
But I only know $\theta \approx h/d$!

Breaking the geometrical degeneracy (CMB plus late-time observations)

Answers:
Roughly 7m away (luckily!)
Roughly 3m tall (really?)

Next lecture

6 December, 9:00-9:50

Measuring the Hubble constant
The Hubble tension

Distance ladder, inverse distance ladder, calibrators, tensions, and so on!