Course webpages:
http://xes.fysik.su.se/HT2016/fmm.html

Mathematical Methods for Physicists
Hand-in problems I

This first hand-in is **due September 9th, 2016 before 15:15 pm**. This is a hard deadline since we want to have the possibility to discuss hand-in problems at the lecture. We will not accept hand-ins after this deadline. You can either give your solutions to Lars at the start of the lecture or mail nice LaTeX solutions to Sunny (*sunny.vagnozzi@fysik.su.se*) directly. Scanned solutions will be accepted if, and only if, the quality is good enough to be read and correct. Please submit solutions in electronic form in pdf-format, only.
Good luck with the problems!

1. Consider the linear homogeneous differential equation

$$\frac{dv}{dt} = -a\,v\,(1+t^2)\,,$$

where $v : \mathbb{R} \to \mathbb{R}$ is a real function depending on $t \in \mathbb{R}$ and $a > 0$ a constant. The variables are separable in this case. When the variables are separated, the equation is exact. Solve this differential equation subject to the initial condition $v(t=0) =: v_0 > 0$ by the following methods:

   (a) separating variables and integrating;

   (b) using the result for a linear homogeneous differential equation derived in the lecture.

2. The equation of motion for a falling object with air resistance can be written as

$$m\,\frac{dv}{dt} = m\,g - b\,v^2$$

where the vertical velocity v is a non-negative function of time t, m denotes the mass of the object, g the (constant) gravitational acceleration, and b the air resistance coefficient (*cf.* example 7.2.1 in the book, this is problem 7.2.16 therein).

   (a) Solve the equation of motion for a parachutist's velocity, assuming that the parachute opens when the parachutist's velocity has reached $v(t=0) = 90\,\mathrm{km/h}$. Show that your solution satisfies the equation of motion and draw a schematic $v(t) - t$-diagram.

   (b) Assuming a "free" fall air resistance coefficient (without parachute) of $b = 0.25\,\mathrm{kg/m}$ and a mass $m = 80\,\mathrm{kg}$, calculate the limiting velocity v_∞.

3. Consider the differential equation for a driven one-dimensional harmonic oscillator with mass M

$$\left(\frac{d^2}{dt^2} + \alpha^2\right) x(t) = \frac{f(t)}{M}\,,$$

where $x(t)$ is a function describing the amplitude at time t, $f(t) := C\,\sin(\beta\,t)$ is the driving force, and $\alpha, \beta > 0$ and C are real constants.

   (a) One solution is $x_1(t) = \cos(\alpha\,t)$. Use the equation[1]

$$x_2(t) = x_1(t) \int^t \frac{\exp\left(-\int^{t'} P(t'')dt''\right)}{x_1^2(t')} dt'$$

from the lecture (or the book) to show that a second linearly independent solution is $x_2(t) = \frac{1}{\alpha}\sin(\alpha\,t)$.

[1] We are using the notation from the book here: The lower limits in the integrals have been omitted since they contribute a term equal to a constant times the first solution which we are not interested in.

(b) To find the particular solution of the inhomogeneous equation, use the formula[1]

$$x_p(t) = x_2(t) \int^t \frac{x_1(t')\,F(t')\,dt'}{\mathcal{W}\{x_1(t'), x_2(t')\}} - x_1(t) \int^t \frac{x_2(t')\,F(t')\,dt'}{\mathcal{W}\{x_1(t'), x_2(t')\}}\,,$$

where $\mathcal{W}\{\cdot,\cdot\}$ denotes the Wronskian determinant.
Show that the particular solution obtained satisfies the differential equation and interpret your results briefly from the physical point of view.

4. Using the Frobenius approach, obtain the general solution of the following equation:

$$3x\,y'' + (3x+1)\,y' + y = 0\,,$$

where y is a function of x, $y' \equiv \frac{dy}{dx}$ and $y'' \equiv \frac{d^2y}{dx^2}$. If the solution can not be represented by an analytical formula, you should give only the four largest terms in the Frobenius expansion when $x \to 0$.