Mathematical Methods for Physicists
Hand-in problems II

Due September 14, 2016 before 15:00. Email your nice LaTeX solutions to Sunny (*sunny.vagnozzi@fysik.su.se*). Scanned solutions will be accepted if, and only if, the quality is good enough to be read and corrected. Please submit solutions in electronic form in the pdf-format, only.
Good luck with the problems!

1. Write the stationary Schrödinger equation

$$-\frac{\hbar^2}{2\,m}\nabla^2\,\Psi + V\,\Psi = E\,\Psi$$

in spherical coordinates, explicitly ($\hbar$, m and E are constants, $V = -\frac{1}{r}$ is a function of the distance to the origin, only). The solution can be written as $\Psi(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$. Find the solution for $\Phi(\phi)$ using separation of variables and set up the separated equations for the $R(r)$ and $\Theta(\theta)$ dependence.

2. Solve the one-dimensional heat equation

$$\frac{\partial\psi}{\partial t} = a^2\frac{\partial^2\psi}{\partial x^2}$$

for the temperature field $\psi(x,t)$ for a rod of length L, with position on the rod given by the variable x, with the two ends of the rod at $x = 0$ and $x = L$ kept at the respective temperatures $\psi(x = 0, t) = 3$ and $\psi(x = L, t) = 0$ for all times t. Initially, $\psi(x, t = 0) = 0$ for $0 \le x \le L$.

Hints: Try to find a general solution on the form $\psi(x,t) = u(x) + v(x,t)$, where the function $u(x)$ satisfies the stationary one-dimensional heat equation and initial conditions $u(0) = 3$ and $u(L) = 0$. Function $v(x,t)$ satisfies the time-dependent one-dimensional heat equation and initial conditions $v(0,t) = 0$, $v(L,t) = 0$ and $v(x,0) = -u(x)$.

3. Let $\mathcal{L}$ be a differential operator and G the corresponding Green's function.
Find the Green's function for

(a)

$$\mathcal{L}\,y(x) := \frac{d^2y(x)}{dx^2} + y(x)\,,$$

with $y(0) = 0$ and $y'(1) = 1$.

(b)

$$\mathcal{L}\,y(x) := \frac{d}{dx}\left(x\frac{dy(x)}{dx}\right) - \frac{n^2}{x}y(x)\,,$$

with $y(0)$ finite and $y(1) = 0$.

4. Consider the differential equation for the driven harmonic oscillator

$$\frac{d^2}{dt^2}y(t) + \omega^2 y(t) = f(t)\,,$$

with initial conditions $y_0 = 0$, $\dot{y}_0 = 0$, where $f(t)$ is some given forcing function. We rewrite

$$f(t) = \int_0^\infty f(t')\delta(t'-t)\,dt'\,,$$

thus decomposing the driving force into an infinite sequence of impulses.

(a) Assuming you know the Green's function $G(t,t')$ satisfying

$$\frac{d^2}{dt^2}G(t,t') + \omega^2 G(t,t') = \delta(t'-t)\,,$$

show that the solution can be written as

$$y(t) = \int_0^\infty G(t,t')f(t')dt'\,.$$

(b) Find the explicit Green's function satisfying the initial conditions ($G = 0$; $\frac{dG}{dt} = 0$ for $t = 0$) and using the above solution, write $y(t)$ for the explicit case $f(t) := \cos(\nu t)$ ($\nu \neq \omega$).