Mathematical Methods for Physicists
Hand-in problems III

Due September 21, 2016 before 15:00. Because Sunny (*sunny.vagnozzi@fysik.su.se*) will be away 19-28 September, physical solutions will **not** be accepted. Please make sure you email your solutions in PDF format, either typed or scanned. Remember to always put your email on top of the paper.

1. Let $z = x + i\,y$ be a complex number with real and imaginary part $x \in \mathbb{R}$ and $y \in \mathbb{R}$, respectively; $u(x, y)$ and $v(x, y)$ functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i \equiv \sqrt{-1}$. Let

$$f(z) = u(x, y) + i\,v(x, y)$$

be an analytic function $f : \mathbb{C} \to \mathbb{C}$.

(a) List the fundamental properties of analytic functions.

(b) Find $f(z)$ if

i. $v(x, y) = 2x\,;$

ii. $v(x, y) = \cos(x)\sinh(y)\,.$

(c) Show that the imaginary part v of any analytic function f solves the Laplace equation $\Delta v = 0$.

2. Explicitly evaluate the complex contour integral

$$\int_\Gamma (z^2 + 1)\,dz\ .$$

Write down appropriate parameterizations of Γ for each of the following contours connecting the point $z_1 = -i$ with $z_2 = 1$. Then use them to evaluate the integral.

(a) Γ is a simple line segment (i.e. straight line);

(b) Γ consists of two simple line segments from $z_1 = -i$ to $z_3 = 0$ and from $z_3 = 0$ to $z_2 = 1$;

(c) Γ is a circular arc, i.e. $z = e^{it}$ where $-\pi/2 \leq t \leq 0$

Give an explanation of these results based on Cauchy's integral theorem.

3. (a) Show that the complex contour integral

$$\frac{1}{2\pi i}\oint_C dz\ z^{m-n-1}$$

where the contour encircles the origin once counter-clockwise, gives 0 if $m \neq n$ and 1 if $m = n$, i.e. it is a representation of the Kronecker delta, δ_{mn}.

(b) Evaluate the complex contour integral

$$\oint_C \frac{z}{z(2z+1)}dz$$

with C the contour unit circle.