Mathematical Methods for Physicists
Hand-in problems III

Due September 20, 2017 before 15:00. This is a hard deadline since we want to have the possibility to discuss hand-in problems at the lecture. We will not accept hand-ins after this deadline. Email your nice LaTeX solutions to Sunny (*sunny.vagnozzi@fysik.su.se*). **Remember to write your email address on the top of the first page of your solutions**! Scanned solutions will be accepted if, and only if, the quality is good enough to be read and corrected. Please submit solutions in electronic form in PDF format only.

[**Total: 10 points**]

1. Let $z = x + i\,y$ be a complex number with real and imaginary part $x \in \mathbb{R}$ and $y \in \mathbb{R}$, respectively; $u(x, y)$ and $v(x, y)$ functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i \equiv \sqrt{-1}$. Let

$$f(z) = u(x, y) + i\,v(x, y)$$

be an analytic function $f : \mathbb{C} \to \mathbb{C}$.

- List the fundamental properties of analytic functions.

- Find $f(z)$ if :
  - $v(x, y) = 2x$.
  - $u(x, y) = \frac{x}{x^2+y^2}$;

  Remember to express f as a function of $z = x + iy$!

-
  - Is the difference of two harmonic functions a harmonic function itself?
  - What about the product of two harmonic functions?

  Note: if you suspect the answer to one of the above is "NO", it's sufficient to show me a counterexample to prove your argument.

- Now consider the function:

$$f(z) = 2x^3y^2 + 3ix^2y^3 ,$$

  where as usual $z = x + i\,y$ is a complex number with real and imaginary parts $x \in \mathbb{R}$ and $y \in \mathbb{R}$, respectively. Show that $f(z)$ is holomorphic only at points lying along the real or imaginary axes.
  [**1+1+1+1+1+1 points**]

2. Explicitly evaluate the complex contour integral:

$$\int_\Gamma (z^2 + 1)\,dz \ .$$

Write down appropriate parameterizations of Γ for each of the following contours connecting the point $z_1 = -i$ with $z_2 = 1$. Then use them to evaluate the integral.

- Γ is a simple line segment (i.e. straight line);
- Γ consists of two simple line segments from $z_1 = -i$ to $z_3 = 0$ and from $z_3 = 0$ to $z_2 = 1$;
- Γ is a circular arc, i.e. $z = e^{it}$ where $-\pi/2 \le t \le 0$

Give an explanation of these results based on Cauchy's integral theorem.
[**1+1+1 points**]

3. i^i:

- Does it make sense to talk about i^i? If not, why? If yes, can you come up with a sensible operative definition of what it means to raise a complex number to the power of another complex number?
  [Hint: you have surely heard about complex logarithms in class. Plus, Euler's formula tells you what it means to raise e to a complex power.]
- How many possible values can i^i take? Should it surprise you that raising a number to the power of another number can give you more than one value?
  [Hint: this can happen with real numbers to the power of a real number as well]
- Finally, give an expression summarizing all possible values of i^i. If you were to come up with a definition of what the principal value of i^i is, what would your choice be?

[**1 point**]