Mathematical Methods for Physicists
Hand-in problems VV

Due October 4, 2017 before 15:00. This is a hard deadline since we want to have the possibility to discuss hand-in problems at the lecture. We will not accept hand-ins after this deadline. Email your nice LaTeX solutions to Sunny (*sunny.vagnozzi@fysik.su.se*). **Remember to write your email address on the top of the first page of your solutions**! Scanned solutions will be accepted if, and only if, the quality is good enough to be read and corrected. Please submit solutions in electronic form in PDF format only.

[**Total: 12 points+bonus**]

1. Evaluate the following integral:

$$\frac{1}{\Gamma\left(\frac{12}{11}\right)}\int_0^\infty dx\ e^{-11x^{11}}\,. \tag{1}$$

Optional question: use your mathematical intuition to guess what the value of the integral $\int_0^\infty dx\ e^{-nx^n}$ is, for $n > 0$ integer.
[Hint: make a wise variable substitution in order to get something which looks like the Γ-function.]
[**3 points**]

2. Many differential equations occur in practice that are not of the standard form but whose solutions can be written in terms of Bessel functions. It can be shown that the differential equation:

$$\frac{d^2y}{dx^2}+\frac{1-2\alpha}{x}\frac{dy}{dx}+\left[\beta^2\gamma^2x^{2\gamma-2}+\frac{\alpha^2-n^2\gamma^2}{x^2}\right]y(x)=0\,,$$

has the solution:

$$y = x^\alpha Z_n(\beta x^\gamma)\,,$$

with $Z_n(\cdot)$ being the Bessel or Neumann function, $J_n(\cdot)$ or $Y_n(\cdot)$ (whether or not the Neumann function is needed depends on whether n is an integer), or any linear combination of them. α, β, γ, and n are constants. Show that, for $\alpha = 0$ and $\beta = \gamma = 1$, this solution makes sense given what you know about Bessel functions. Then, solve the differential equation:

$$x^2\frac{d^2y}{dx^2}-19x\frac{dy}{dx}+(x^2+84)y=0\,. \tag{2}$$

[**3 points**]

3. Solve the following differential equation:

$$x^2\frac{d^2y}{dx^2}+(2x^2+x)\frac{dy}{dx}+(2x^2+x)y(x)=0\,. \tag{3}$$

[Hint: the substitution $u = y(x)e^x$ might come in handy]
[**3 points**]

4. Consider the Bessel function of the first kind $J_\nu(x)$ and the Hankel functions (also known as Bessel function of the third kind) $H_\nu^{1,2}(x)$, defined from the Bessel functions of the first and second kind (the latter also known as Neumann functions). Prove the following Wronskian relations:

- $J_\nu(x)\frac{dH_\nu^{(1)}}{dx}-\frac{dJ_\nu(x)}{dx}H_\nu^{(1)}(x)=\frac{2i}{\pi x}$.

- $J_{\nu-1}(x)H_{\nu}^{(1)}(x) - J_{\nu}(x)H_{\nu-1}^{(1)}(x) = \frac{2}{i\pi x}$.

[**3 points**]

5. Bonus exercise. **WARNING**: this one is very hard, and beyond what is done in the course! Only attempt it if you have finished the other four, in other words don't waste your time on it if you haven't finished the rest. Moreover, points will only be attributed if the exercise is fully solved (i.e. it will either be full or no points). You've been warned, now the exercise goes:
Consider the following equation:

$$\mathcal{A}(y) \equiv \frac{1}{2} \frac{\int_0^\infty dx \; \frac{x^{\frac{y-2}{2}}}{e^{\sqrt{x}}-1}}{\int_0^\infty dx \; x^{y-1} e^{-x}} = 0\,, \tag{4}$$

with $\Re(y) > 1$, where $\Re$ denotes the real part of a complex number, but one can analytically continue the function for any y, so long as $\Re(y) \neq 1$, which is what we will assume in the rest of this exercise. The above equation admits infinitely many solutions. Let us refer to them as y_i. Now, consider the subset of y_i, which we will call $\tilde{y}_i$, such that $\Im(\tilde{y}_i) \neq 0$, where $\Im$ denotes the imaginary part of a complex number. Show that $-\pi\Re(\tilde{y}_i) = \mathcal{P}(i \ln i)$, where $\mathcal{P}(i \ln i)$ denotes the principal value of $i \ln i$ (recall the principal value of a complex logarithm is defined by restricting its imaginary part to $(0, 2\pi]$).
[Hint: try to make a variable substitution in the numerator of $\mathcal{A}(y)$ in order to get something which resembles (or coincides with) Eq.(13.62) in Arfken.]