Mathematical Methods for Physicists
Hand-in problems VI

Due October 12, 2016 before 15:00. It is preferable if you email your solutions to Sunny (*sunny.vagnozzi@fysik.su.se*), either LaTeX'd or scanned (PDF format), or alternatively bring them in person to A5:1078. Submission through third parties, especially in the case of physical copies, is highly discouraged as there is the risk of them not reaching the intended receiver. Please remember to put your email on top.

[**Total: 12 points**]

1. (a) For $z \in \mathbb{R}$, $m \in \mathbb{N}$, and $J_m(z)$ Bessel functions of the first kind, derive the Jacobi-Anger expansion of a plane wave in a series of cylindrical waves:

$$e^{iz\cos\theta} = \sum_{m=-\infty}^{\infty} i^m J_m(z) e^{im\theta}\,.$$

[**3 points**]

(b) From the above, using $J_{-m}(z) = (-1)^m J_m(z)$, derive the following:

$$\cos(z\cos\theta) = J_0(z) + 2\sum_{m=1}^{\infty}(-1)^m J_{2m}(z)\cos(2m\theta)\,.$$

[**3 points**]

2. Many differential equations occur in practice that are not of the standard form but whose solutions can be written in terms of Bessel functions. It can be shown that the differential equation

$$y'' + \frac{1-2a}{x}y' + \left[\left(bcx^{c-1}\right)^2 + \frac{a^2 - p^2c^2}{x^2}\right]y = 0$$

has the solution

$$y = x^a Z_p(bx^c)\,,$$

with $Z_p(\cdot)$ being the Bessel or Neumann function, $J_p(\cdot)$ or $Y_p(\cdot)$, or any linear combination of them, a, b, c, p are constants.
Use this to find the solutions to the differential equation

$$xy'' + 5y' + xy = 0\,.$$

[**2 points**]

3. The amplitude $U(\rho, \varphi, t)$ of a vibrating circular membrane of radius a satisfies the wave equation

$$\nabla^2 U = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2} = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}$$

Here v is the phase velocity of the wave, determined by the properties of the membrane.

(a) Show that a physically relevant solution is

$$U = (\rho, \varphi, t) = J_m(k\rho)(c_1 e^{im\varphi} + c_2 e^{-im\varphi})(b_1 e^{i\omega} + b_2 e^{-i\omega t})$$

[**3 points**]

(b) From the Dirichlet boundary condition $J_m(ka) = 0$, find the allowed values of k.
[**1 point**]