Mathematical Methods for Physicists
Hand-in problems VII

Due October 19, 2016 before 15:00. Email your solutions to Sunny (*sunny.vagnozzi@fysik.su.se*), either LaTeX'd or scanned (PDF format), or if necessary photographed.

[**Total: 14 points**]

1. Work in spherical polar coordinates and prove the following relation:

$$\frac{\partial}{\partial z}\left[\frac{P_l(\cos\theta)}{r^{l+1}}\right] = -(l+1)\frac{P_{l+1}(\cos\theta)}{r^{l+2}}$$

The above equation is useful to show that the derivative of a multipole can be written in terms of the next higher-order multipole, which is very useful e.g. when studying perturbations to the Boltzmann equation in early Universe cosmology.
[**3 points**]
[*Hint*: $\frac{\partial}{\partial_z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$.]

2. The spherical Bessel functions j_n may be seen as an integral transform of the Legendre polynomials P_n. Show that:

$$j_n(z) = \frac{1}{2}(-i)^n \int_0^\pi d\theta\ e^{iz\cos\theta} P_n(\cos\theta)\sin\theta\,, \tag{1}$$

where $n = 0, 1, 2, 3, ...$ [**4 points**]
[*Hint*: use $j_n(z) = \frac{z^n}{2^{n+1}n!}\int_0^\pi d\theta\ \cos(z\cos\theta)\sin^{2n+1}\theta$]

3. The amplitude of a scattered wave $f(\theta)$ calculated from the Schrödinger equation for a central scattering potential can be obtained by expanding the incident plane wave in terms of Legendre polynomials as shown above. The asymptotic approximation then reads:

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\, e^{i\delta_l}\ \sin(\delta_l)\, P_l(\cos(\theta))\,,$$

where $0 \le \theta \le 2\pi$ is the scattering angle, l is the angular momentum eigenvalue, $\hbar k$ is the incident momentum and δ_l is the phase shift caused by the interaction with the central scattering potential. The total cross section is given by $\sigma_{tot} = \int |f(\theta)|^2\, d\Omega$ with the solid angle Ω. Show that:

$$\sigma_{tot} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\ \sin^2(\delta_l)\,.$$

[**4 points**]

4. Prove that:

$$\frac{1-t^2}{(1-2tx+t^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)t^n$$

[**3 points**]
[*Hint*: Evaluate $2t\frac{\partial g(x,t)}{\partial t} + g(x,t)$, where $g(x,t)$ is the generating function.]