Mathematical Methods for Physicists
Hand-in problems VII

Due October 18, 2017 before 15:00. This is a hard deadline since we want to have the possibility to discuss hand-in problems at the lecture. We will not accept hand-ins after this deadline. Email your nice LaTeX solutions to Sunny (*sunny.vagnozzi@fysik.su.se*). **Remember to write your email address on the top of the first page of your solutions**! Scanned solutions will be accepted if, and only if, the quality is good enough to be read and corrected. Please submit solutions in electronic form in PDF format only.

[**Total: 14 points**]

1. Consider a function $f(x)$ defined on the range $(-1, 1)$. Show that:

$$\int_{-1}^{1} dx\ f^2(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \left[\int_{-1}^{1} dx\ f(x) P_n(x) \right]^2 , \tag{1}$$

where $P_n(x)$ denote the Legendre polynomials.
[**4 points**]
[Hint: try writing the integrand in the left-hand side as a Legendre series expansion. Don't the coefficients in the series look suspiciously like the integrand on the right-hand side?]

2. Prove that:

$$\int_{-1}^{1} dx\ \left(-1 + \frac{45}{8} x + 3x^2 - \frac{105}{4} x^3 + \frac{189}{8} x^5 \right) P_n(x) = 0 \,. \tag{2}$$

for $n \geq 6$, as well as $n = 1$, $n = 3$, and $n = 4$.
[**3 points**]

3. Consider the derivative of the Legendre polynomials, which we will call $\mathcal{P}_n(x) \equiv \frac{dP_n(x)}{dx}$. Prove that:

$$\mathcal{P}_n(1) = \frac{\Gamma(n+2)}{\Gamma(3)\Gamma(n)} , \tag{3}$$

where Γ denotes the Γ-function you have seen in class.
[**4 points**]
[Hint: I don't recommend you write out the Γ function according to its integral representation. Rather, try to think of one of the most remarkable properties of this function. Once you have transformed the right-hand into a nice expression, try proving the relation by induction.]

4. Recall the definition of Fourier cosine transform of a function $f(x)$:

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx\ f(x) \cos(\omega x) , \tag{4}$$

and the corresponding inverse Fourier cosine transform:

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} d\omega\ g(\omega) \cos(\omega x) . \tag{5}$$

Now consider the function $f(x)$ defined by:

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{otherwise} . \end{cases} \tag{6}$$

First of all, find the Fourier cosine transform of this function, $g_c(\omega)$. Then, take the inverse Fourier cosine transform of $g_c(\omega)$ and use your result to show that:

$$f(x) = \frac{2}{\pi} \int_0^\infty d\omega \; \frac{\sin(\omega)\cos(\omega x)}{\omega} \, . \tag{7}$$

[**3 points**]